The Risk of Risk-Sharing: Diversification and Boom-Bust Cycles*

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Abstract

I model a shock whereby financial intermediaries can better diversify borrowers' idiosyncratic risks. A sector-specific diversification improvement induces intermediaries to reallocate funds toward the shocked sector. As lending spreads fall, intermediaries build up leverage over time. The result is a fragile sectoral boom that can end in an economy-wide bust. Among other financial shocks – relaxed borrowing constraints, lower capital requirements, higher risk tolerance, lower uncertainty, and foreign safe-asset demand – none generate both sectoral reallocation and financial leveraging. I apply the model quantitatively to the recent housing cycle. Feeding in a novel mortgage diversification index, the model generates the measured increase in household credit coincident with a 1-2% decline in mortgage spreads. In the subsequent bust, spreads in all sectors spike by 2% as aggregate output drops.

JEL Codes: D14, G11, G12.

Keywords: Diversification, Financial Shocks, Credit Cycles, Housing Boom, Banking, Idiosyncratic Risk, Financial Innovation, Securitization, Deregulation.

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1 Introduction

Many booms feature a sectoral bias – for example, a US "railroad boom" in the 1850s, an "agriculture boom" in the 1880s, and a "housing boom" in the 2000s. Sectoral booms, however, can end in economy-wide busts. The aforementioned booms ended with the "Panic of 1857," the "Panic of 1893," and the "Great Recession." Securitization is a factor connecting these three episodes. In each case, loans to the booming sector were increasingly pooled and sold to investors.[1] Motivated by these observations, this paper offers a financial theory that links sectoral booms to economy-wide busts.

I present a model in which financiers provide funds to two distinct productive sectors. A critical function of financiers' is to *diversify* within-sector idiosyncratic risks, which they accomplish by holding a large portfolio of loans. In practice, financiers diversify by making a variety of loans or holding securitized portfolios.

Suppose financiers' diversification improves within one of the sectors, as might occur with the advent of securitization. In the short run, a *reallocation effect* arises: facing an improved risk-reward trade-off, financiers redirect funds toward the sector with newly improved diversification, which raises sectoral investment. The reallocation effect helps explain why many booms feature a sectoral bias. Indeed, both the boom and its sectoral bias originate from sector-specific diversification improvements.

Meanwhile, better diversification reduces sectoral risk premia. A series of low risk premia earned by financiers results in a redistribution of wealth from financiers toward the rest of the economy. To maintain their funding activities, financiers must borrow more, which I call the *leverage effect*. If financial leverage is destabilizing, the leverage effect explains why a sectoral boom can lead to an economy-wide bust.

I adopt a particular connection between leverage and stability. Assume financiers face a leverage constraint. If the leverage effect is strong enough, financiers endogenously hit their constraint, at which point they must de-lever. A less-qualified type of financier, whom I call distressed investors, purchases financiers' liquidated loan portfolios and serves as the marginal supplier of any new loans. The de-leveraging thus disturbs both

[1] See Riddiough and Thompson (2012) and Calomiris and Schweikart (1991) for an account of the securitization of railroad-adjacent farm loans in the 1850s. See Eichengreen (1984), Snowden (1995), and Snowden (2007) for an account of farm mortgage securitizations in the 1880s. See below for evidence pertaining to the 2000s housing cycle. There is also an emerging notion that 1830s lending to plantations, backed by slaves as collateral, was responsible for a rapid cotton boom, followed by the "Panic of 1837" after the cotton slowdown. Indeed, Matthew Desmond notices the "parallels between the Panic of 1837 and the 2008 financial crisis. All the ingredients are there: mystifying financial instruments that hide risk while connecting bankers, investors and families around the globe..." See https://www.nytimes.com/interactive/2019/08/14/magazine/slavery-capitalism.html

sectors, not only the booming sector. Lending spreads in both sectors rise sharply at the constraint, resembling a financial crisis.

A financial crisis might generate a bust in "real variables" like consumption and investment for several reasons. In the baseline model, I assume the participation of distressed investors triggers deadweight losses. Such losses might be justified by distressed investors' lower productivity in the advisory, monitoring, and screening activities that the financial sector typically provides. With deadweight losses, the de-leveraging episode triggers an inefficient bust. In the appendix, I explore alternatives linking a financial crisis to an economy-wide bust.

Given this inefficiency, why do financial crises occur in equilibrium? The answer is a *risk-taking externality* in financiers' portfolio decisions. When individual financiers take risk, they do not account for the downward pressure they put on risk premia and, by extension, the profitability of other financiers. Lower financier profitability raises the prospect of future binding constraints, and hence a crisis. Financiers privately ignore this socially-undesirable prospect.

The entire cycle is amplified if diversification improves in a *lower-quality* sector, i.e., a sector that is higher-risk or more reliant on external financing. Low-quality borrowers offload more risk onto financier balance sheets, and diversification has a larger marginal benefit when applied to a riskier balance sheet. This generates larger reallocation and leverage effects, meaning a larger and more asymmetric boom, but also a higher chance of a broad bust.

In a quantitative exercise, I apply the model to the recent US housing cycle. I create a novel index of idiosyncratic mortgage banking risk, in order to measure mortgage diversification. My approach has two key advantages relative to the existing empirical literature. First, the index encapsulates deregulation, financial innovations, and mergers, which tend to have similar qualitative effects but are difficult to compare quantitatively.[2] Second, the index has risk-based units, useful for calibrating economic models. Using this index, I extract a time series measure of mortgage diversification, which increased substantially from 1990 to 2006.[3]

[2] For example, deregulations that allowed banks to operate across state borders clearly improved loan portfolio diversification, but how does the magnitude of this improvement compare to the rise of securitization?

[3] I do not measure diversification in non-mortgage lending markets. But several facts suggest that deregulations and securitization were geared primarily towards household finance. First, the results of Rice and Strahan (2010) and Favara and Imbs (2015) together provide causal evidence that bank branching deregulations in the late 1990s and early 2000s disproportionately affected mortgage finance, relative to firm finance. Second, mortgage securitization grew must faster than commercial loan securitization: the ratio of outstanding mortgage securities to corporate securities grew by 50% from 1990-2006. See Appendix F.1.

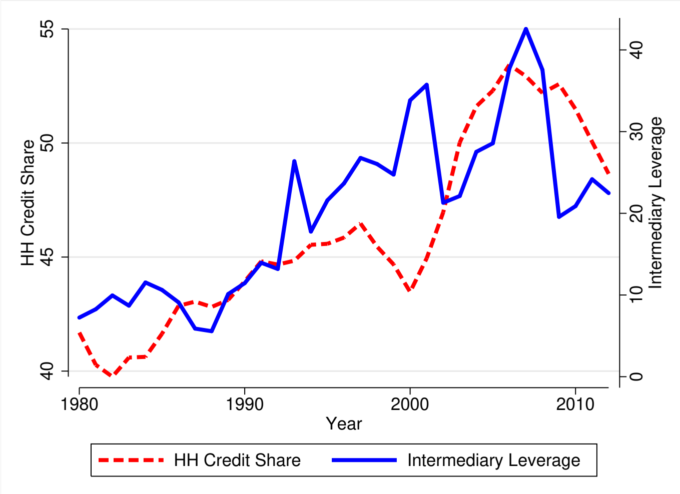

Figure 1: "HH Credit Share" denotes households' share of total non-financial corporate credit, from the Flow of Funds; "Intermediary Leverage" denotes broker-dealer leverage, from Adrian et al. (2014).

Figure 1 shows the 1990-2006 increase in diversification is correlated with the reallocation and leverage effects. Reallocation is proxied by the household credit share, and leverage is proxied by broker-dealers' assets-to-equity ratio. Inserting my diversification time series into the calibrated model, I match the household credit share in figure 1 and the 2% drop in mortgage rates documented in the literature.[4] Model-implied financier leverage also rises in the boom, qualitatively in line with figure 1. Because leverage constraints start binding in the bust, financiers' implied funding costs increase by over 2%, in line with data on financial crises.[5] The credit spreads of both sectors, not just housing, spike by the same magnitude of financiers' funding costs. In a counterfactual exercise without diversification improvements, there is no episode resembling a financial crisis, with spikes in funding costs or credit spreads.

While diversification improvements trigger a cycle characterized by sectoral reallocation and financier leveraging, other financial shocks might do the same. Motivated by the literature, I study five other financial shocks in the model – a loan-to-value shock,[6] a capital-requirement shock,[7] a risk-tolerance shock,[8] an uncertainty shock,[9] and a foreign-savings shock.[10] Among these, none generate both reallocation and leverage. The core

[4] See Justiniano et al. (2017).

[5] See Fleckenstein and Longstaff (2018).

[6] LTV-type shocks are studied by Jermann and Quadrini (2012), Kiyotaki et al. (2011), Justiniano et al. (2015b), Favilukis et al. (2017), and Greenwald (2016).

[7] This resembles the relaxation of banks' "lending constraints" in Justiniano et al. (2015a).

[8] For example, Kindleberger (1978) says, "The monetary history of the last four hundred years has been replete with financial crises. The pattern was that investor optimism increased as economies expanded, the rate of growth of credit increased and economic growth accelerated, and an increasing number of individuals began to invest for short-term capital gains..." See Kaplan et al. (2017) for an analysis of optimism shocks on housing markets.

[9] See Di Tella (2017) for a model of intermediation with idiosyncratic volatility shocks.

[10] See the "global savings glut" hypothesis of Bernanke (2005) and Favilukis et al. (2017) for a model.

rationale for this result is, unlike other shocks, a diversification improvement differentially impacts one sector and differentially improves financiers' investment-opportunity set relative to other agents'. This intuition applies to any multi-sector intermediated macroeconomy.

This paper contributes to three literatures: (1) the literature on the effects of financial intermediation on the macroeconomy; (2) the literature on diversification and other financial shocks; and (3) the literature on the recent housing cycle.

By focusing on the financial sector, my framework shares many features with the "financial accelerator" literature on macroeconomic dynamics with financial frictions. Net worth of borrowers, producers, or financiers acts as a buffer to fundamental economic shocks in these models, building off of insights by Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). Like He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014), I employ continuous-time methods, to extend these ideas to study crisis dynamics and other nonlinearities.

My model's structure differs from this literature differ in three aspects. First, a key role of financiers in my model is to diversify idiosyncratic risks. Most of this literature studies financial intermediaries who are more productive investors but in fact less diversified than other agents. Second, I include two sectors, to study financiers' reallocation between them. Third, I study diversification improvements, a type of financial shock, which lead to interesting and different boom-bust dynamics. Most financial-accelerator papers focus on standard fundamental shocks, which are amplified by the endogenous concentration of risk in the financial sector. These models generate a bust only after a long sequence of negative fundamental shocks, because intermediaries are well-capitalized following a boom.[11]

By contrast, my economy can experience an endogenous bust which is then amplified by small negative fundamental shocks, because of the "leverage effect" and the presence of the financier leverage constraint. The leverage effect, whereby diversification lowers fundamental risk but is offset by higher risk-taking, relates to the "Peltzman effect" in automobile safety (Peltzman, 1975) or the "volatility paradox" in macro-finance (Brunnermeier and Sannikov, 2014).[12] The leverage constraint means that booms characterized

[11]Some models not based on net worth can generate busts following few adverse shocks. For example, Boissay et al. (2016) present a model based on information asymmetries that generates a finance-centric boom-bust cycle. Gorton and Ordoñez (2016) generates cycles based on the interaction between real productivity and financiers' incentives to accept collateral.

[12]Demsetz and Strahan (1997) document this effect empirically for larger bank holding companies, whose better diversification is offset by increased risk-taking. Adrian et al. (2015) argue the negative correlation between financial sector ROE and expected returns supports the profitability mechanism in my paper. Also related, Wagner (2008, 2010, 2011) develops a series of theoretical models to illustrate

by rising financial leverage can destabilize the economy, as in Minsky (1992).[13]

The theoretical possibility that financial innovation may be inefficient is well-known (Hart, 1975). In my paper, better financier diversification is a form of financial innovation. Welfare can decrease through exacerbation of a pecuniary externality, whereby individual financiers do not internalize how their risk-taking decisions reduce the total equity of the financial system (Acharya et al., 2017).

My approach to modeling diversification is related to the model in Gârleanu et al. (2015), which uses a Brownian bridge on a circle of locations to model correlated shocks. Investors allocate funds along arcs on the circle, which prevents full diversification of idiosyncratic risks. I use the theory of Gaussian processes to develop a new stochastic process, which I call a *Brownian cylinder*, that maintains cross-sectional correlations on a circle but accommodates an infinite-horizon, continuous-time setting. This apparatus could be useful in other settings where continuous-time methods are fruitful (e.g., optimal stopping problems, occasionally-binding portfolio constraints, heterogeneous-agent macro models).

In my quantitative analysis, I apply the framework to the recent US housing boom. Motivating this exercise is a large empirical literature arguing credit-supply increases were the key driver of the boom.[14] For example, Favara and Imbs (2015) study the effect of credit on house prices using bank-branching deregulations of the late 1990s and early 2000s as a credit-supply instrument. The deregulations plausibly allowed banks to achieve better-diversified loan portfolios. My paper argues better mortgage diversification is an important credit-supply shock driving the boom and bust.

Regarding the bust, most of this quantitative literature incorporating financial shocks generates a bust only after applying a negative financial shock (e.g., Favilukis et al. (2017) and Kaplan et al. (2017) generate busts with constraint and pessimism shocks). Missing

downsides of financial diversification. Closest to my paper is Wagner (2008), in which the banking sector features a risk-taking externality. A banks does not internalize that high-risk, low-liquidity portfolio choices increase other banks' probability of inefficient liquidation. Better diversification improves risk-reward trade-offs, thereby worsening the externality.

[13] For example, Minsky (1992) says, "...Over periods of prolonged prosperity, the economy transits from financial relations that make for a stable system to financial relations that make for an unstable system."

[14] Adelino et al. (2012) use a regression-discontinuity design to show that (conforming) mortgage securitization reduces lending rates and raises house prices. Mian and Sufi (2018) provide causal evidence that the abrupt increase in private-label MBS activity dramatically increased house prices and led to the bust. See also Mian and Sufi (2009), Mian and Sufi (2011), and Di Maggio and Kermani (2017). This credit-supply view is not uncontroversial. For example, Haughwout et al. (2011), Chinco and Mayer (2015), and Albanesi et al. (2017) point to housing investors (who mortgage multiple properties) as a driver of the 2000s boom-bust cycle. Adelino et al. (2016) argue that mortgage credit increased proportionally in all income groups. These findings are at odds with the traditional rationing-based view, in which lower-quality borrowers are the largest beneficiaries of the credit-supply increase.

from this literature is the idea that the nature of the boom can make the subsequent bust more likely and larger. Specifically, my model differs in that a diversification-induced boom creates financial instability.

Even under the narrative of a deterioration of household balance sheets (Mian and Sufi, 2015), the timing and extreme severity of the Great Recession are puzzling. Bernanke (2018) uses such arguments to conclude that the housing bust is not enough; the 2008 financial panic is key to a coherent narrative. My paper provides a model taking this finance view.

The paper is organized as follows. Section 2 studies the reallocation and leverage effects. Section 3 extends the model to allow for financial crises. Section 4 quantifies diversification improvements in the recent US housing cycle. Section 5 concludes. Appendix A details construction of the Brownian cylinder. Appendix B contains microfoundations. Appendix C contains equilibrium derivations and proofs. Appendix D analyzes other financial shocks. Appendix E contains model extensions. Appendix F contains empirical analysis.

2 Two-Sector Model: Reallocation and Leverage

The model of this section is meant to introduce the primary mechanisms of my framework: reallocation and leverage. I introduce two sectors that produce with their own capital stocks. I will show that an increase in diversification of one sector's risks leads to reallocation towards that sector and, in the long run, to increased overall leverage by financial intermediaries.

2.1 Setup

Time is continuous $t \geq 0$. The model features two groups of agents: insiders and financiers. Insiders are additionally split into two groups, depending on which of two productive sectors they inhabit, A or B. These insiders invest in capital, and consume. To finance their capital purchases, insiders issue outside securities and put up some of their own net worth. These outside securities are held by financial intermediaries, which are operated by financiers. To finance their investment activities, financiers use their own net worth as well as risk-free debt. Agents in each group are indexed by $i \in [0, 1]$, which will represent an agent's location, to be described below.

Preferences. Agents have logarithmic utility over the single consumption good:

$$\mathcal{U}_t := \mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(s-t)}\log(c_s)ds\Big]. \tag{1}$$

Locations and Idiosyncratic Risk. Agents are arranged on a circle, which has locations indexed by $i \in [0,1]$. Locations will be special because they feature different idiosyncratic shocks. These shocks directly hit the evolution of productive capital. Mathematically, capital held by an insider at location i evolves as

$$dk^A_{i,t} = dI^A_{i,t} + k^A_{i,t}\hat{\sigma}_A dW^A_{i,t} \tag{2}$$
$$dk^B_{i,t} = dI^B_{i,t} + k^B_{i,t}\hat{\sigma}_B dW^B_{i,t}. \tag{3}$$

In (2)-(3), dI^A, dI^B are desired investment, and W^A, W^B are idiosyncratic shocks (more on these processes below). There are no aggregate shocks for now.

For two reasons, I assume no investment adjustment costs, as in Cox et al. (1985). First, my focus is on incomplete financial markets rather than investment frictions. A minimal number of frictions affords maximum theoretical clarity, and my results on boom-bust cycles must be attributed to the financing frictions. Second, zero adjustment costs allows me to obtain analytical solutions to the equilibrium of this economy.

I assume the idiosyncratic shocks $W^A_{i,t}$ and $W^B_{i,t}$ are independent copies of a stochastic process with the following properties.

Assumption 1 (Shocks). *Assume the following for $W := \{W_{i,t} : i \in [0,1], t \geq 0\}$.*

(i) Fixing a location $i \in [0,1]$, $W_{i,t}$ is a standard Brownian motion.

(ii) For any two locations $i, j \in [0,1]$, the shock correlation is

$$corr(dW_{i,t}, dW_{j,t}) = 1 - 6dist(i,j)(1 - dist(i,j)), \tag{4}$$

where $dist(i,j) := \min(|i-j|, 1-|i-j|)$ is a distance metric on the circle of circumference 1.

(iii) $W_{i,t}$ is continuous in (i,t) almost-surely, under the Euclidean distance metric on the cylinder $\widetilde{dist}((i,s),(j,t)) := [|s-t|^2 + dist(i,j)^2]^{1/2}$.

Given part (i) of Assumption 1, the increment $dW_{i,t}$ is iid over time, for fixed location i. Part (ii) of Assumption 1 means the shock correlations between locations decrease with their distance from one another.[15] Nearby locations have nearly perfect shock correlation. Two locations that are far away from one another have negative correlation. Does any such stochastic process exist?

Lemma 2.1. *A stochastic process exists which satisfies Assumption 1.*

The key step in Lemma 2.1 is proving W can be constructed as a Gaussian process with the appropriate covariance function, which needs to be symmetric and positive semi-definite. Because W evolves on a circle over time, which looks like a cylinder, I call it a *Brownian cylinder*.

With Assumption 1, we can establish some distributional properties of the Brownian cylinder, in particular that it contributes no aggregate risk.

Lemma 2.2. *Under Assumption 1, there is no aggregate risk, i.e., $\int_0^1 (dW_{i,t})di = 0$ almost-surely. More generally, the local variance of a unit investment divided amongst the shocks along an arc of length Δ is equal to $(1-\Delta)^2$, i.e.,*[16]

$$Var_t\Big(\int_i^{i+\Delta} \Delta^{-1} dW_{j,t} dj\Big) = (1-\Delta)^2 dt.$$

Consequently, $W_{i,t}^{\Delta} := (1-\Delta)^{-1}\Delta^{-1}\int_i^{i+\Delta} W_{j,t}dj$ is a standard Brownian motion.

Given Lemma 2.2, the shock $dW_{i,t}$ is correlated across locations but washes out in the aggregate, the sense in which it is idiosyncratic. The surprising part of this result is that we only needed to specify the covariance structure of the shocks, and this property alone pins down the integral of all the shocks. Figure 2 plots a simulation of the Brownian cylinder.[17]

Asset Markets. Sectoral capital is homogeneous, which implies the location-invariant unit prices $q_{A,t}$ and $q_{B,t}$. With zero adjustment costs, we will have $q_{A,t} \equiv q_{B,t} \equiv 1$ in equilibrium. Finally, there is a zero-net-supply riskless bond market that returns $r_t dt$. All agents can access this bond market frictionlessly.

[15] This shock correlation owes to Gârleanu et al. (2015). Using the Brownian bridge on a "circle," they construct discrete-time idiosyncratic shocks that are cross-sectionally correlated but contain zero aggregate risk. They find the dividend correlation is exactly $1 - 6\text{dist}(i,j)(1-\text{dist}(i,j))$. My proof of Lemma 2.1 would apply for any appropriate correlation function $v(i,j)$ that depends only on $\text{dist}(i,j)$ (i.e., stationary correlation function).

[16] My notation convention is that "$i+\Delta$" represents $i+\Delta-\lfloor i+\Delta \rfloor$ when indexing a position on the circle.

[17] Because W is a Gaussian process, a simulation can be obtained by drawing a normal random vector with the appropriate covariance matrix.

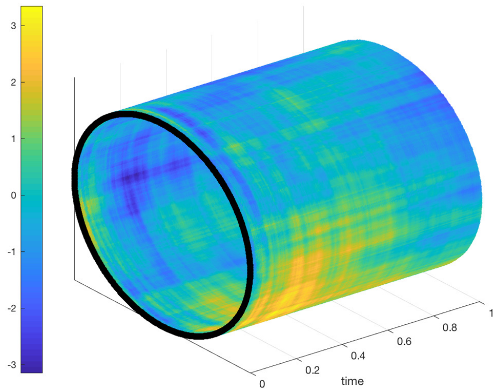

Figure 2: One shock realization of the Brownian cylinder $\{W_{i,t} : t \in [0,1]\}$, for i at 500 evenly spaced locations. Each cross-section of the cylinder is the circle of locations. Colors represent the size of $W_{i,t}/\sqrt{t}$.

Insider Problem. Because of symmetry between the two sectors and their insiders, I describe the problem of an insider in generic sector $z \in \{A, B\}$. On the asset side, insiders hold capital in their firm, which produces according to an "AK" technology. The representative insider at location i produces $G_z k^z_{i,t}$ for $z \in \{A, B\}$. Due to the absence of adjustment costs, the return on capital is given by

$$dR^z_{i,t} = G_z dt + \hat{\sigma}_z dW^z_{i,t}, \quad z \in \{A, B\}. \tag{5}$$

Insiders are also marginal in the risk-free debt market, at the interest rate r_t. On the liability side, insiders can obtain funding from financiers against their capital, by signing a contract promising the return of

$$d\tilde{R}^z_{i,t} := (r_t + s^z_{i,t})dt + \big(dR^z_{i,t} - \mathbb{E}_t[dR^z_{i,t}]\big), \quad z \in \{A, B\}.$$

This liability is a way for insiders to shed some of the idiosyncratic risk associated with production. The "spread" charged by financial intermediaries is given by $s^z_{i,t}$.

I assume insiders finance a fixed fraction ϕ_z of the value of their enterprise from financiers in the form of outside equity, paying $\phi_z k^z_{i,t} d\tilde{R}^z_{i,t}$ to financiers. In Appendix B.1, I demonstrate that this risk-sharing arrangement is an optimal solution to a standard moral-hazard problem.

Combining the assumptions above, insider net worth $n^z_{i,t}$ evolves as

$$dn^z_{i,t} = \underbrace{(n^z_{i,t} r_t - c^z_{i,t})dt}_{\text{consumption-savings}} + \underbrace{k^z_{i,t}(dR^z_{i,t} - r_t dt)}_{\text{capital ownership}} - \underbrace{\phi_z k^z_{i,t}(d\tilde{R}^z_{i,t} - r_t dt)}_{\text{outside funding}}, \quad z \in \{A, B\}. \tag{6}$$

Given the inability to frictionlessly trade the idiosyncratic risk of capital, one can think of the differential $k^z_{i,t}\mathbb{E}_t[dR^z_{i,t} - r_t dt - \phi_z(d\tilde{R}^z_{i,t} - r_t dt)]$ as a compensation for idiosyncratic risk. Mathematically, households solve

$$\max_{n_i^z, c_i^z, k_i^z} \mathcal{U}_{i,t}^z, \quad z \in \{A, B\}, \tag{7}$$

subject to (6), $n_{i,t}^z \geq 0$, $k_{i,t}^z \geq 0$, where $\mathcal{U}_{i,t}^z$ is given by the logarithmic utility functional (1).

Financier Problem. Financiers serve a diversification and safe-asset-creation role. Financiers hold a partially-diversified portfolio of equity in each of the two sectors. They fund these activities by borrowing in riskless debt and using their own net worth.

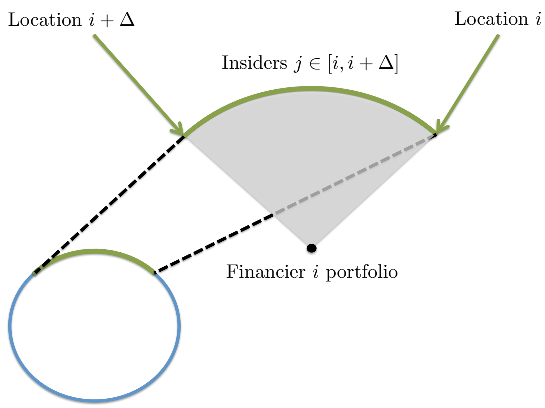

Figure 3: Circle of locations and financiers' partially diversified portfolios. Financiers have potentially different diversification parameters Δ_A and Δ_B for each sector.

I model diversification as follows. Financiers are tied to locations, just as insiders are. A financier located at $i \in [0, 1]$ invests in a portfolio of insiders' securities located "nearby" in the sense that they lie in a connected interval adjacent to location i. Define $\Delta_z \in [0, 1]$ to be the length of this interval for insiders in sector $z \in \{A, B\}$. Insiders financed by finanicer i are those with $j \in [i, i + \Delta_z] \mod [0, 1]$. Here, the Δ_z are exogenously fixed numbers, not choices by financiers. This partial but imperfect diversification arc on the circle may be visualized in figure 3.

For simplicity, I assume financiers fund all insiders within their investment arc symmetrically. Let $\lambda_{i,t}^z$ represent location-i financiers' funding, per unit of their net worth, of sector-z insiders. In other words, location-i financiers supply $\lambda_{i,t}^z \Delta_z^{-1} n_{i,t}^F$ of funds to each sector-z insider $j \in [i, i + \Delta_z] \mod [0, 1]$, rather than allowing $\lambda_{i,t}^z$ to also vary by destination.[18]

[18]Relaxing this assumption does not change the results significantly.

Putting everything together, the financier's net worth evolves dynamically as follows:

$$dn_{i,t}^F = \underbrace{(n_{i,t}^F r_t - c_{i,t}^F)dt}_{\text{consumption-savings}} + \underbrace{\lambda_{i,t}^A n_{i,t}^F \Delta_A^{-1} \int_i^{i+\Delta_A} (d\tilde{R}_{j,t}^A - r_t dt)dj}_{\text{funding portfolio (sector } A)} + \underbrace{\lambda_{i,t}^B n_{i,t}^F \Delta_B^{-1} \int_i^{i+\Delta_B} (d\tilde{R}_{j,t}^B - r_t dt)dj}_{\text{funding portfolio (sector } B)}. \tag{8}$$

Financiers solve

$$\max_{n_i^F, c_i^F, \lambda_i^A, \lambda_i^B} \mathcal{U}_{i,t}^F \tag{9}$$

subject to (8), $n_{i,t}^F \geq 0$, $\lambda_{i,t}^A \geq 0$, $\lambda_{i,t}^B \geq 0$, where $\mathcal{U}_{i,t}^F$ is given by (1).

Free Mobility. At this point, I make an important technical assumption that keeps the equilibrium construction tractable. Specifically, I assume a free-mobility condition between locations, which allows us to study a symmetric equilibrium.

Assumption 2 (Mobility)**.** *Insiders and financiers are freely mobile among locations i.*

Under Assumption 2, idiosyncratic shocks will wash out in aggregate, but their presence matters for individual behavior. A similar free-mobility assumption has been used across the idiosyncratic "islands" of Gertler and Kiyotaki (2010).

2.2 Equilibrium

Definition 1. *An* equilibrium *consists of price and allocation processes, adapted to the shocks $\{(W_{i,t}^A, W_{i,t}^B) : i \in [0,1], t \geq 0\}$, such that all agents solve their optimization problems and all markets clear. Prices consist of the interest rate r_t and spreads $s_{i,t}^A, s_{i,t}^B$. Allocations consist of capital and equity holdings $(k_{i,t}^A, k_{i,t}^B, \lambda_{i,t}^A, \lambda_{i,t}^B)$, and consumption choices $(c_{i,t}^A, c_{i,t}^B, c_{i,t}^F)$. A* symmetric equilibrium *is an equilibrium in which all objects are independent of i for each t. The market-clearing conditions are as follows.*

$$\begin{aligned}
&\textit{(Goods market)} \quad \int_0^1 [G_A k_{i,t}^A + G_B k_{i,t}^B] di dt = \int_0^1 [c_{i,t}^A + c_{i,t}^B + c_{i,t}^F] di dt + \int_0^1 [dI_{i,t}^A + dI_{i,t}^B] di \\
&\textit{(Bond market)} \quad \int_0^1 [n_{i,t}^A + n_{i,t}^B + n_{i,t}^F] di = \int_0^1 [k_{i,t}^A + k_{i,t}^B] di \\
&\textit{(Funding markets)} \quad \int_{i-\Delta_z}^{i} \Delta_z^{-1} \lambda_{j,t}^z n_{j,t}^F dj = \phi_z k_{i,t}^z, \quad \forall i \in [0,1], \quad z \in \{A, B\}.
\end{aligned}$$

In this paper, I analyze a symmetric equilibrium, in which locations are exactly identical in their net worths. Such a construction is feasible (and weakly optimal) under free-mobility. Studying this equilibrium allows me to avoid keeping track of the full distribution of wealth among locations, which would otherwise be necessary to know the evolution of aggregates.[19]

For construction of the symmetric equilibrium, define aggregate capital $K_t := \int_0^1 [k^A_{i,t} + k^B_{i,t}]di$ and the capital distribution $\kappa_t := K_t^{-1} \int_0^1 k^A_{i,t} di$. Define the wealth shares

$$\alpha_t := \frac{N_{A,t}}{N_{A,t} + N_{B,t}} \quad \text{and} \quad \eta_t := \frac{N_{F,t}}{N_{F,t} + N_{A,t} + N_{B,t}}$$

where $N_{A,t} := \int_0^1 n^A_{i,t} di$, $N_{B,t} := \int_0^1 n^B_{i,t} di$, and $N_{F,t} := \int_0^1 n^F_{i,t} di$ are aggregate net worths. The only state variables in a symmetric equilibrium will be (α_t, η_t, K_t). Therefore, in what follows, I drop location i subscripts from all variables whenever the meaning is clear. All stationary variables will be solely functions of (α_t, η_t), whereas growing variables grow with K_t. State dynamics are $dK_t = K_t \iota_t dt$, $d\alpha_t = \mu^\alpha_t dt$, and $d\eta_t = \mu^\eta_t dt$, where the aggregate investment rate ι_t is determined from $\iota_t K_t dt := dI^A_t + dI^B_t$.[20] The equilibrium is computed explicitly.

Proposition 2.3 (Two-Sector Equilibrium)**.** *Let Assumptions 1 and 2 hold. Then, there exists a unique symmetric equilibrium with state variables (α, η). The state dynamics are*

$$\mu^\alpha = \alpha(1-\alpha)[\hat{\pi}_A^2 - \hat{\pi}_B^2] \tag{10}$$
$$\mu^\eta = \eta(1-\eta)[\hat{\pi}_{F\to A}^2 + \hat{\pi}_{F\to B}^2 - \alpha\hat{\pi}_A^2 - (1-\alpha)\hat{\pi}_B^2], \tag{11}$$

where

$$\hat{\pi}_A := \frac{\kappa(1-\phi_A)\hat{\sigma}_A}{\alpha(1-\eta)} \quad \textit{and} \quad \hat{\pi}_B := \frac{(1-\kappa)(1-\phi_B)\hat{\sigma}_B}{(1-\alpha)(1-\eta)} \tag{12}$$
$$\hat{\pi}_{F\to A} := \frac{\kappa\phi_A(1-\Delta_A)\hat{\sigma}_A}{\eta} \quad \textit{and} \quad \hat{\pi}_{F\to B} := \frac{(1-\kappa)\phi_B(1-\Delta_B)\hat{\sigma}_B}{\eta} \tag{13}$$

[19] For details on equilibrium of a model with correlated shocks and limited mobility, see Khorrami (2018), in which the distribution of net worth across locations becomes a state variable.

[20] Note that, although sectoral investment dI^A_t and dI^B_t will not, in general, be absolutely continuous with respect to time (Lebesgue measure), the sum must be absolutely continuous as a consequence of goods market clearing.

are shadow idiosyncratic risk prices. The capital distribution is given by

$$\kappa = \min(1, \max(0, \tilde{\kappa})),$$
$$\text{where} \quad \tilde{\kappa} := \frac{G_A - G_B + \left[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta}\right]\hat{\sigma}_B^2}{\left[\frac{(1-\phi_A)^2}{\alpha(1-\eta)} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta}\right]\hat{\sigma}_A^2 + \left[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta}\right]\hat{\sigma}_B^2}. \tag{14}$$

I should clarify a few elements of the equilibrium in Proposition 2.3. First, even though the two sectors produce the same consumption good, each sector can receive a non-trivial allocation of resources because of their risk properties. Indeed, the sectoral shocks W^A and W^B are independent, so it is efficient to diversify these shocks by producing some output in each sector. The qualitative insights below survive in a model with differentiated goods, which provides an additional rationale for production diversification. See Appendix E.1 for the model with Cobb-Douglas preferences over the consumption goods.

Second, the expected excess return on each capital stock can be decomposed into idiosyncratic risk premia earned by insiders and financiers. These idiosyncratic risk premia are non-trivial due to imperfect diversification by both insiders (who must hold $1-\phi$ fraction of their capital risk) and financiers (who can only diversify Δ fraction of the locations). Indeed, for sectors $z \in \{A, B\}$,

$$\underbrace{G_z - r}_{\text{total risk premium}} = \underbrace{(1-\phi_z)\hat{\sigma}_z\hat{\pi}_z}_{\text{insiders' idio risk premium}} + \underbrace{\phi_z(1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}}_{\text{financiers' idio risk premium}}. \tag{15}$$

Indeed, $(1-\phi_z)\hat{\sigma}_z$ and $\phi_z(1-\Delta_z)\hat{\sigma}_z$ represent the quantity of idiosyncratic risk held by insiders and financiers, respectively, and $\hat{\pi}_z$ and $\hat{\pi}_{F\to z}$ are the prices of these risks. These idiosyncratic risk prices measure the marginal utility response to a negative idiosyncratic shock.

Finally, the economy is deterministic in aggregate and approaches "steady state" as $t \to \infty$. See Proposition C.1.

The equilibrium can be conveyed graphically. The left panel of figure 4 plots the supply and demand in sector A's lending market, with the idiosyncratic risk price $\hat{\pi}_{F\to A}$ against the financier portfolio λ^A (there is a symmetric graph for sector B). This is a novel feature of this paper: in most models, financiers are perfectly diversified, trivially eliminating financiers' idiosyncratic risk prices as an equilibrating price, e.g., $\hat{\pi}_{F\to A} = 0$.

The increasing line is funding supply: financiers' optimal portfolio λ^A is simply a mean-variance portfolio trading off idiosyncratic risk compensation, $\hat{\pi}_{F\to A}$, against the

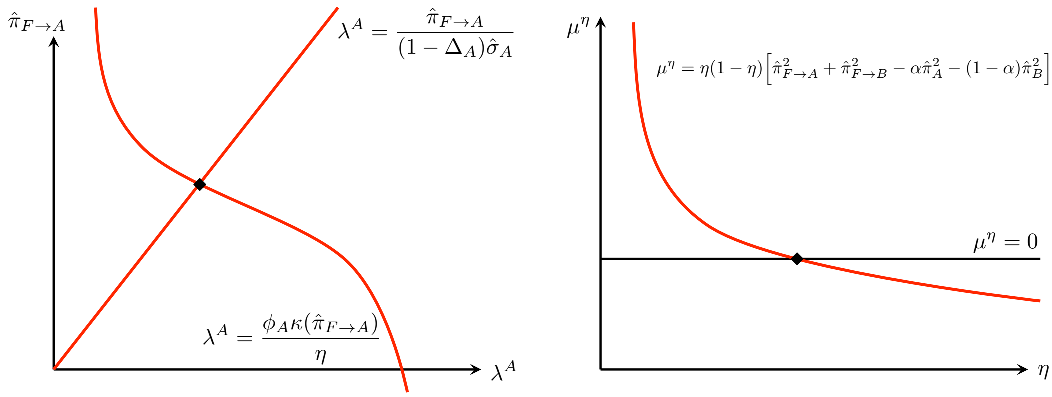

Figure 4: Steady-state equilibrium.

idiosyncratic volatility of the portfolio, $(1-\Delta_A)\hat{\sigma}_A$.

The downward-sloping curve plots funding demand, which is constructed from insiders' optimal capital choice. Sector A capital demand, relative to aggregate capital, is

$$\kappa = \frac{G_A - r - \phi_A s_A}{(1-\phi_A)^2 \hat{\sigma}_A^2}(1-\eta)\alpha.$$

Insiders retain $(1-\phi_A)$ of their capital risk as inside equity and optimally trade off its variance, $(1-\phi_A)^2\hat{\sigma}_A^2$, against its expected return. Inside equity earns the expected excess return $G_A - r$ on capital, net of the lending spread s_A paid to financiers on ϕ_A of outside equity. Because the spread s_A is fair, it compensates financiers for idiosyncratic risk, i.e., $s_A = (1-\Delta_A)\hat{\sigma}_A \hat{\pi}_{F\to A}$. A higher risk price $\hat{\pi}_{F\to A}$ increases spreads s_A and lowers capital demand κ. Lower capital demand reduces funding demand through the equity-market-clearing relationship $\phi_A \kappa = \lambda^A \eta$, which is the downward-sloping curve plotted in figure 4.[21]

The right panel shows the dynamics of η. The drift μ^η balances the relative profitabilities of financiers and insiders, which are governed by their idiosyncratic risk prices:

$$\mu^\eta = \eta(1-\eta)\Big[\underbrace{\hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}}_{\text{financier profitability}} - \underbrace{\left(\alpha\hat{\pi}_A + (1-\alpha)\hat{\pi}_B\right)}_{\text{insider profitability}}\Big].$$

[21] To get an expression for the downward-sloping curve, take the difference between asset pricing equation (15) for $z = A, B$ and substitute $\hat{\pi}_A$ and $\hat{\pi}_B$ to get the following:

$$\kappa = (1-\eta)\Big[\frac{(1-\phi_A)^2\hat{\sigma}_A^2}{\alpha} + \frac{(1-\phi_B)^2\hat{\sigma}_B^2}{1-\alpha}\Big]^{-1}\Big[G_A - G_B - \left[\phi_A s_A - \phi_B s_B\right] + (1-\eta)^{-1}\frac{(1-\phi_B)^2\hat{\sigma}_B^2}{1-\alpha}\Big].$$

Notice that, holding s_B fixed, κ is decreasing in s_A, hence $\hat{\pi}_{F\to A}$.

As a function of η, μ^η is typically decreasing, because $\hat{\pi}_{F\to A}$ and $\hat{\pi}_{F\to B}$ are decreasing in η, whereas $\hat{\pi}_A$ and $\hat{\pi}_B$ are increasing in η. This downward-sloping property is why the economy converges to the steady state.

2.3 Diversification Improvements

In this section, I illustrate the reallocation and leverage effects discussed in the introduction. To do this, I construct a time-path, or an "impulse response function" (IRF), for all variables following a change in diversification. See Appendix C.2 for more details on these IRFs. Due to the tractability offered by logarithmic utility and frictionless physical investment, computing IRFs is not problematic.

Broadly speaking, there are three types of diversification IRFs. The first type of IRF treats diversification changes as *unanticipated*, in the sense that economic agents perceive zero probability of diversification improvements, even though improvements repeatedly occur. A second type of IRF treats the diversification changes as *fully anticipated*, in the sense that news about the future diversification path breaks at time τ, and after that time, agents know the entire future time-path of diversification. A third type of IRF treats diversification shocks as *partially anticipated*, in the sense that agents know diversification levels follow a particular stochastic process. As shown by the lemma below, these three types of IRFs are equivalent in this model.

Lemma 2.4. *Suppose any of the following situations hold:*

(i) ***Unanticipated:*** $(\Delta_{A,t}, \Delta_{B,t})$ *follows an arbitrary stochastic process, but agents place zero probability on any future changes.*

(ii) ***Fully Anticipated:*** $(\Delta_{A,t}, \Delta_{B,t})$ *follows a deterministic path. At time* τ*, agents are informed about a new future path* $\{(\Delta_{A,t}, \Delta_{B,t}) : t \geq \tau\}$.

(iii) ***Partially Anticipated:*** $(\Delta_{A,t}, \Delta_{B,t})$ *follows an arbitrary Itô process. All agents are unconstrained in markets for Arrow claims on the shocks* $d\Delta_{z,t} - \mathbb{E}_t[d\Delta_{z,t}]$.

Then, the economy is in the equilibrium of Proposition 2.3 with $(\Delta_{A,t}, \Delta_{B,t})$ *representing* (Δ_A, Δ_B) *at every point in time* t.

This lemma relies on the optimally-myopic behavior of log utility agents, who care only about the current level of diversification, not its future probability distribution. In addition, IRF computation is simplified due to a no-jump property.

Lemma 2.5. *Consider an unanticipated time-*t *shock to* (Δ_A, Δ_B)*. Then,* $(\alpha_t, \eta_t) = (\alpha_{t-}, \eta_{t-})$.

The intuition for Lemma 2.5 is that portfolio holdings are pre-determined before a shock, so wealth can only jump if asset prices jump. But frictionless investment implies capital prices are always equal to one; in particular, they cannot jump. Thus, this model contains no "impact response" to diversification shocks.

With this equivalence proved, suppose diversification improves in sector A, i.e., $\Delta_A \uparrow$. Figure 5 illustrates the adjustment to the new steady state.

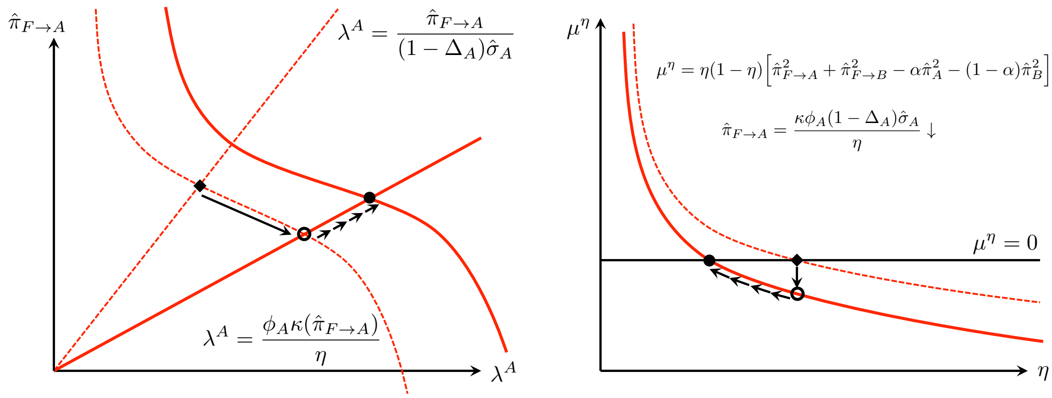

Figure 5: Equilibrium before and after an increase in diversification, $\Delta_A \uparrow$.

In the short run, better diversification increases sectoral funding supply because it improves financiers' risk-reward trade-off. Graphically, this improvement is captured by the outward rotation of the supply curve (left panel), which results in a shift from the diamond to the hollow circle.[22] This shift reduces equilibrium risk compensation $\hat{\pi}_{F\to A}$ and generates a discontinuous increase in the sector A capital share κ. Although aggregate capital will never jump in equilibrium, its sectoral allocation can, due to frictionless investment.

This short-run outcome is the *reallocation effect*. Diversification-induced reallocation can partly explain the fact that sectoral capital shares are negatively correlated with sectoral risk premia, documented by Bansal et al. (2017). Reallocation can also occur with an increase in TFP G_A. But as equation (15) shows, productivity-based reallocation must raise the sectoral risk premium, exactly as in Cochrane et al. (2007).

Lower risk prices $\hat{\pi}_{F\to A}$ reduce financier profitability, so the drift μ^η shifts downwards (right panel). Over time, η drifts down. Financiers are happy to decumulate,

[22] Note that there is a small outward shift in the supply (flattening of the slope) on impact because, as footnote 21 shows, κ is a decreasing function of $s_A = (1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}$, so it depends on Δ_A independently of $\hat{\pi}_{F\to A}$. This issue is eliminated if we use s_A rather than $\hat{\pi}_{F\to A}$ as the relevant price. In that case, $\lambda^{A,\text{supply}} = s_A(1-\Delta_A)^{-2}\hat{\sigma}_A^{-2}$ represents the supply curve and $\lambda^{A,\text{demand}} = \phi_A\kappa(s_A)\eta^{-1}$ the demand curve, where $\kappa(\cdot)$ depends on Δ_A only through s_A.

because a lower quantity of idiosyncratic risk necessitates a lower precautionary savings buffer. However, lower relative wealth means financiers must accumulate leverage to continue their scale of financing operations. This dynamic effect is captured by the gradual outward shift in the demand curve (left panel), which results in a shift from the hollow circle to the solid circle. Because financiers are present in both sectors, a similar effect occurs in sector B.

I call this dynamic force the *leverage effect*. Financier leverage (assets/equity) is

$$\text{leverage} := \lambda^A + \lambda^B = \frac{\phi_A \kappa + \phi_B(1-\kappa)}{\eta}, \tag{16}$$

so declines in η tend to raise leverage. The next result formalizes this analysis.

Proposition 2.6 (Reallocation/Leverage)**.** *Suppose at time t the economy is sufficiently close to an interior steady state and then there is a small increase in Δ_A. The sector-A capital share increases, $\kappa_t > \kappa_{t-}$. Sufficient conditions for $\mu_t^\eta < \mu_{t-}^\eta$ are (i) nearly symmetric sectors; or (ii) $|\log(\kappa_t/\kappa_{t-})|$ is not too large.*

Figure 6 translates the graphical analysis of figure 5 to a time-path. The top panels show responses to a one-time unanticipated Δ_A-shock at time $t = 0$, when the system is in steady state. The bottom panels show responses to a gradual increase in Δ_A. Performing this experiment often raises the question of how to interpret several increases to Δ_A; in particular, what probability do agents attach to future diversification increases? Given the equivalence result of Lemma 2.4, we are able to sidestep this question. The left panels show time-paths for Δ_A. The middle and right panels illustrate the responses of κ_t and $\lambda_F^A + \lambda_F^B$ – the reallocation and leverage effects.

These time-paths connect the model to the 1990s-2000s housing boom. As shown in figure 1, this episode featured a large sectoral reallocation from corporate credit to household credit and a rise in financial intermediary leverage. Thus, if we interpret sector A as housing and sector B as productive capital, a gradual increase in Δ_A, corresponding to rising mortgage securitization or gradual banking deregulation, can match these qualitative patterns.

2.4 Endogenous Credit Standards as Financier Leverage

The diversification-induced leverage build-up ("leverage effect") springs from a reduction in financier profitability, through lower equilibrium spreads. Empirically, there is

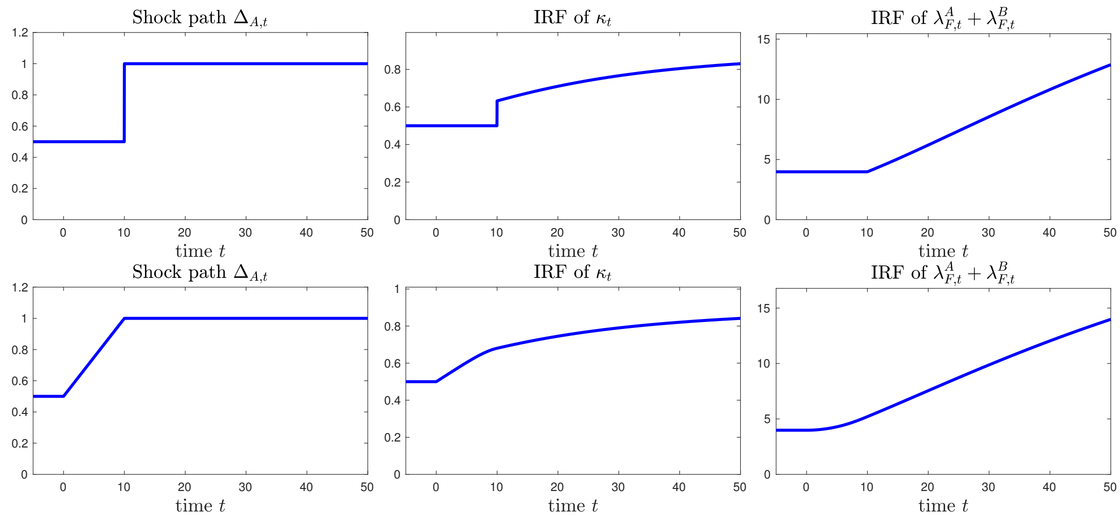

Figure 6: IRFs to a one-time shock (top panels) and gradual increase (bottom panels) from $\Delta_A = 0.5$ to $\Delta_A = 1$ at time $t = 0$. In this example, $G_A = G_B = 0.1$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $\rho = 0.02$, and $\Delta_B = 0.5$.

some support for this during the 2000s housing boom, as mortgage spreads fell and commercial bank profitability saw a modest decline.[23]

Theoretically, the leverage build-up is a general response to better diversification and can be observed even in absence of a profitability decline. For example, better diversification initially lowers lending spreads, so insiders may optimally borrow more by increasing their outside funding (e.g., ϕ_A). Greater outside funding and better diversification have opposing effects on equilibrium spreads, so financier profitability may increase or decrease. At the same time, higher ϕ_A directly raises financiers' assets/equity ratio by equation (16). Intuitively, if credit quantity (rather than spreads) is the main margin of adjustment, bank leverage will still increase.

To analyze such a situation, Appendix B.2 generalizes the moral-hazard problem of insiders to generate the possibility of time-varying ϕ_A and ϕ_B. In this setup, the moral-hazard problem is smoothed in such a way that optimal short-term contracts cannot eliminate agency costs. Optimal issuance ϕ_A equates the marginal diversification benefits from offloading risk (arising because financiers are better-diversified than insiders) to marginal moral-hazard costs (arising because insiders will divert more resources when they keep less skin in the game). Improved financier diversification increases the marginal benefit of issuance, so ϕ_A rises with Δ_A. Although a lower skin-in-the-game requirement exacerbates insiders' agency problem, now-better-diversified financiers tolerate this cost. Credit standards are optimally relaxed, analogous to the story "securi-

[23]Justiniano et al. (2017) show spreads declined; Appendix F.1 shows commercial bank profits.

tization led to lax screening" in Keys et al. (2010). Thus, endogenizing credit standards demonstrates the leverage effect does not rely on falling financier profitability.[24]

2.5 Comparison to Other Financial Shocks

Because of the model's tractability, it is possible to study several other "financial shocks". In Appendix D, the model is extended to study an LTV shock, a capital-requirement shock, a risk-tolerance shock, an uncertainty shock, and a foreign-savings shock – all of which have been linked by the extant literature to boom-bust cycles, most recently related to the 2000s US housing boom (see the relevant citations in Appendix D). Other than the diversification shock, none can produce both a sectoral reallocation and financier leveraging in my model. These results are formalized in Propositions D.1-D.5 and summarized in Table 1.

Financial Shocks	**Stylized Facts** Sectoral Reallocation	Financial Leverage
Diversification	$+$	$+$
LTV	$+$	$\sim$
Capital requirements	$\sim$	$+$
Financier risk-tolerance	$\sim$	$+$
Insider risk-tolerance	$+$	$-$
Idiosyncratic sectoral risk	$+$	$\sim$
Foreign savings	$\sim$	$+$

Table 1: Stylized facts and financial shocks. "$+$" indicates a positive response in the stylized fact to the referenced shock. "$-$" indicates a negative response in the stylized fact to the shock. "$\sim$" indicates a neutral or ambiguous response in the stylized fact to the shock.

The basic intuition for the results of Table 1 is that reallocation and leverage are generated by a sector-specific credit supply shock, i.e., a shock that reduces financiers' cost of lending and is biased towards a particular sector. The other shocks considered are either not biased towards a particular sector (capital requirements, financier risk-tolerance, foreign savings / safe-asset holdings), or they affect credit demand as much or more than credit supply (LTV, insider risk-tolerance, idiosyncratic risk).

[24]The time-dynamics of financier leverage are ambiguous in the literal version of this story, because of the opposing effects of higher Δ_A and higher ϕ_A on financier profitability. An extension with slow adjustment of ϕ_A (perhaps through within-sector heterogeneity) seems promising to induce a slow build-up of financier leverage.

3 Diversification-Induced Financial Crises

The model of Section 2 illustrates the reallocation and leverage mechanisms in a simple way. The key shortcoming of that model is the absence of financial fragility: even though diversification reduces the financier wealth share, no additional macro-financial risk emerges (and one can show the equilibrium is efficient). Below, I modify the model by introducing a cost to high leverage, with a leverage constraint and an imperfect deleveraging technology, which allows for fire sales and financial crises. An alternative extension, whereby there is a flight-to-safety episode at the leverage constraint, is solved in Appendix E.2.

3.1 New Features

Aggregate Risk. To the capital evolution (2)-(3), I add aggregate shocks:

$$dk_{i,t}^z = dI_{i,t}^z + k_{i,t}^z[\hat{\sigma}_z dW_{i,t}^A + \sigma_z \cdot dZ_t], \quad z \in \{A, B\},$$

where $Z := (Z^A, Z^B)$. For simplicity, I assume orthogonal shocks: $\sigma_A \cdot \sigma_B = 0$.

There is also a continuously-settled, zero-net-supply futures market for trading claims directly on aggregate risk. Investing one unit of net worth in this claim earns the excess return $\pi_t dt + dZ_t$, where π_t is the market price of risk associated with the dZ_t shocks. I assume financiers and distressed may trade in this market unconstrained, but insiders cannot. In reality, insiders of firms may be prevented from market trading due to incentive problems. This assumption generates stochastic fluctuations, because aggregate risk cannot be shared perfectly.

Leverage Constraints. Most importantly, I introduce a financier leverage constraint:

$$\lambda_{F,t}^A + \lambda_{F,t}^B \leq \bar{\lambda}, \qquad \bar{\lambda} > 1. \tag{17}$$

Borrowing constraints like (17) can be a reduced-form for financier default costs that rise sharply with high leverage, or they may arise due to incentive problems.[25] In Appendix B.3, I micro-found the constraint with a simple agency problem. There, the endogenously-determined maximal leverage $\bar{\lambda}$ is a function of the economic state and model parameters but is crucially not increasing in diversification Δ_A, Δ_B.

[25] See, for example, Kehoe and Levine (1993), Hart and Moore (1994), Kocherlakota (1996), Kiyotaki and Moore (1997), Holmstrom and Tirole (1997), Gertler and Kiyotaki (2010), and Di Tella and Sannikov (2016).

The leverage constraint modifies portfolio choices by introducing an auxiliary variable (Lagrange multiplier) that I denote ζ_t. Thus, financiers' optimal portfolios are given by

$$\lambda_F^A = \frac{(s_A - \sigma_A \cdot \pi - \zeta)^+}{(1-\Delta_A)^2 \hat{\sigma}_A^2} \quad \text{and} \quad \lambda_F^B = \frac{(s_B - \sigma_B \cdot \pi - \zeta)^+}{(1-\Delta_B)^2 \hat{\sigma}_B^2} \tag{18}$$

The standard complementary slackness determines when leverage constraints bind:

$$0 = \min\left\{\zeta,\, \bar{\lambda} - \lambda_F^A - \lambda_F^B\right\}. \tag{19}$$

The portfolio choices (18) are simple because the constraints (leverage and shorting) are homogeneous in wealth, and because all agents have log utility. See Appendix B.4 for a derivation using convex duality methods as in Cvitanić and Karatzas (1992) and Garleanu and Pedersen (2011). The presence of ζ helps us understand that a binding leverage constraint works similarly to a rise in intermediary funding costs. Indeed, as (18) suggests, equilibrium with a leverage constraint is identical to an unconstrained economy in which financiers perceive funding costs of $r + \zeta$ rather than r. Hence, I will sometimes refer to ζ as the "shadow funding cost."

Distressed Investors. I introduce a fourth category of agent, which I call "distressed investors," who may also extend financing to insiders, but are less qualified to do so. In particular, for each unit of financing, distressed investors must pay a pecuniary cost χ out of their returns. Such costs may be a reduced-form for search costs, information-acquisition costs, fundraising costs, etc.[26] I implicitly assume these activities take time and other resources that would otherwise be used in production, so that this pecuniary cost is a deadweight loss to the economy. Although they are less skilled lenders, distressed investors do not face the leverage constraint (17). Finally, for quantitative purposes below, mainly to control average financier leverage, I assume financiers can have a higher discount rate, $\rho_F \geq \rho$, than distressed investors and insiders. Otherwise, distressed investors are identical to financiers.

With distressed investors, equilibrium requires that we keep track of distressed investors' aggregate net worth $N_{D,t}$. In symmetric equilibrium, the wealth distribution is now characterized by three state variables:

$$\alpha := \frac{N_A}{N_A + N_B}, \quad \eta := \frac{N_F + N_D}{K}, \quad \text{and} \quad x := \frac{N_F}{N_F + N_D}.$$

[26]Distressed investors may also be interpreted as the marginal buyers of loans and securities on the secondary market, after the first-best financiers begin defaulting. For example, Chernenko et al. (2016) show mutual fund holdings of ABS grew in the financial crisis. Relatedly, He et al. (2010) study ABS holdings of hedge funds, commercial banks, and investment banks.

Competition among insiders ensures distressed investors must also charge spreads s_A and s_B. Consequently, their return-on-investment is given by

$$\Delta_A^{-1} \int_i^{i+\Delta_A} (d\tilde{R}_{j,t}^A - \chi dt) dj + \Delta_B^{-1} \int_i^{i+\Delta_B} (d\tilde{R}_{j,t}^B - \chi dt) dj.$$

Their portfolio choices are given by

$$\lambda_D^A = \frac{(s_A - \sigma_A \cdot \pi - \chi)^+}{(1-\Delta_A)^2 \hat{\sigma}_A^2} \quad \text{and} \quad \lambda_D^B = \frac{(s_B - \sigma_B \cdot \pi - \chi)^+}{(1-\Delta_B)^2 \hat{\sigma}_B^2}. \tag{20}$$

Financial distress is said to occur when either spread rises beyond χ, such that $\lambda_D^A + \lambda_D^B > 0$.

Because the participation cost is modeled as a pecuniary cost, any equilibrium financial distress leads to inefficiency. The costs of financial distress appear in the modified resource constraint:

$$\underbrace{\iota + x\eta\rho_F + (1 - x\eta)\rho}_{\text{Investment + Consumption}} = \underbrace{\kappa G_A + (1-\kappa) G_B}_{\text{Output}} - \underbrace{\chi\eta(1-x)(\lambda_D^A + \lambda_D^B)}_{\text{Costs of Financial Distress}}. \tag{21}$$

These costs are mechanically tied to periods of distress, although they scale with the degree of distress, i.e., the size of the costs depends on the level of participation by distressed investors.

Overlapping Generations. Lastly, I introduce a "perpetual youth" overlapping-generations (OLG) structure, to ensure a generically stationary wealth distribution, similar to Gârleanu and Panageas (2015). All agents perish independently at the Poisson rate δ. Since this assumption augments all agents' subjective discount rate by $+\delta$, parameters ρ and ρ_F should be thought of as inclusive of δ (see Lemma C.2). There are no markets to hedge idiosyncratic death shocks. To keep the population size constant, newborns arrive at the same rate. Among newborns, the fraction entering sector z is ν_z, with $\nu_A + \nu_B + \nu_F + \nu_D = 1$. Dying agents' wealth is redistributed equally to newborns.

3.2 Equilibrium and Distress

With these new features, equilibrium is derived (in closed form) in Appendix C.3. The first finding links financial distress to leverage constraints, which are, in some sense, both necessary and sufficient.

Proposition 3.1 (Distress with Leverage Constraints)**.** *In equilibrium:*

(i) If $\phi_A = \phi_B$, then $\lambda^A_{F,t} + \lambda^B_{F,t} = \bar{\lambda}$ implies $\lambda^A_{D,t} + \lambda^B_{D,t} > 0$.

(ii) Suppose $\chi \geq \bar{\lambda} \max\{(1-\Delta_A)^2\hat{\sigma}_A^2, (1-\Delta_B)^2\hat{\sigma}_B^2\}$. Then, $\lambda^A_{F,t} + \lambda^B_{F,t} < \bar{\lambda}$ implies $\lambda^A_{D,t} + \lambda^B_{D,t} = 0$.

Part (i) of Proposition 3.1 is a case in which distressed investors participate whenever (17) binds. Intuitively, because financiers are unable to raise new equity, they will be *forced* to de-lever upon hitting constraint (17), independent of their risk exposures and degree of hedging activities. De-leveraging automatically results in inefficient participation by distressed investors. In this sense, leverage constraints are sufficient for financial distress.

Part (ii) says that distress requires binding leverage constraints, under certain parameterizations. This can be understood as follows. Distressed investors enter the market in the earlier of two situations: (a) to allow financiers to offload some idiosyncratic risk (i.e., for optimal risk-sharing); or (b) because financiers are leverage-constrained. If financiers do not need to bear an extreme amount of idiosyncratic risk (e.g., if Δ_A, Δ_B are high enough after a diversification improvement), then distress must come from binding constraints. For more discussion on this point, see Appendix C.4, which also contains the proof of Proposition 3.1.

Figure 7 illustrates these features. When η and x are low, financiers hit their leverage constraints. In this region, *financial distress* emerges as distressed investors enter the market and begin lending. Financial distress generates a *jump* in the spreads of both sectors, even in sector B where $\Delta_B = 1$.

Why is there a discontinuous jump in spreads upon financial distress? For rational forward-looking financiers, why not withhold some investment just outside the distressed region, to avoid hitting their leverage constraint as aggregate dynamics move the economy into a region with much higher premia? My financiers optimally have zero such hedging demands. Crucially, to obtain enough funds to avoid leverage constraints in the distress region, financiers would need to consume less for an extended period of time before financial distress. Under log utility, the trade-off between this loss and the gain from the higher returns in the distressed region is exactly balanced by the myopic portfolio (18).

3.3 Dynamics: Endogenous Busts and Financial Instability

This section shows how "endogenous busts" and "financial instability" can follow a diversification improvement in this new environment.

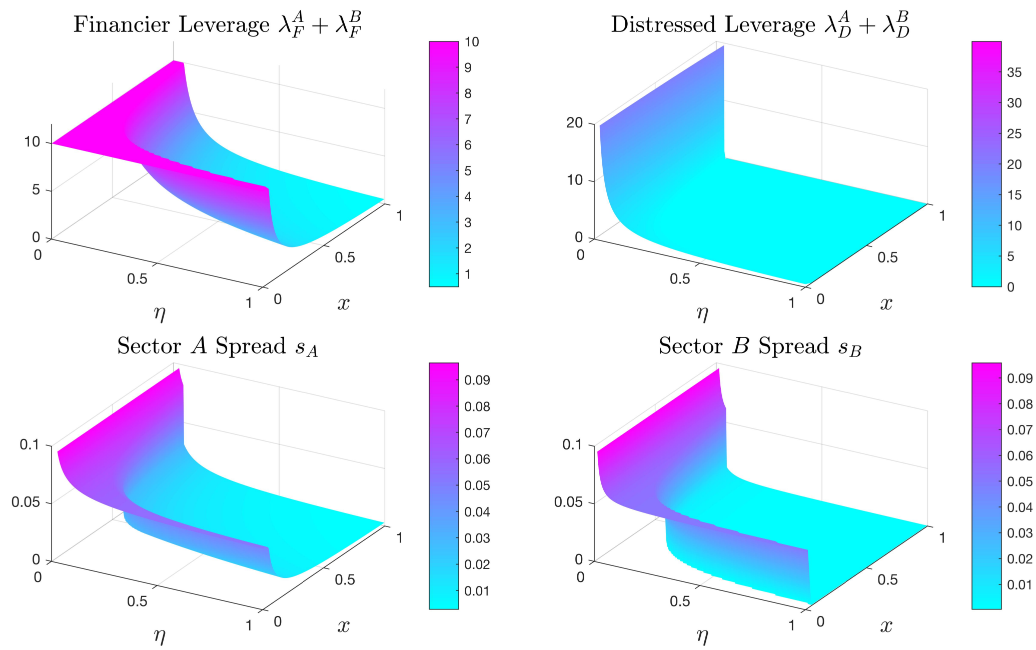

Figure 7: Equilibrium functions of (η, x) with $\alpha = 0.5$ fixed. Parameters: $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $G_A = G_B = 0.1$, $\Delta_A = 0.5$, $\Delta_B = 1$, $\rho = 0.02$, $\rho_F = 0.06$, $\chi = 0.05$, and $\bar{\lambda} = 10$.

Definition 2. *Suppose the economy is in stationary equilibrium, and there is a one-time shock at time τ. We say an* endogenous bust *occurs if there is a predicted future decrease in investment or consumption, relative to capital, i.e., if there exists $t > 0$ such that $\mathbb{E}_{\tau-}[Y_{\tau+t}] < Y_{\tau-}$, where $Y := \iota + x\eta\rho_F + (1 - x\eta)\rho$.*

Notice, from the resource constraint (21), investment and consumption are equal to "endogenous productivity" $\kappa G_A + (1-\kappa)G_B$ net of "distress costs" $\chi\eta(1-x)(\lambda_D^A + \lambda_D^B)$. For theoretical clarity, I will focus on a deterministic environment and an economy with symmetric sectors (in particular, $G_A = G_B$), so that the endogenous productivity term is constant.

Proposition 3.2. *Consider the equilibrium of Proposition C.3, with symmetric sectors and deterministic dynamics, i.e., $G_A = G_B \equiv G$, $\phi_A = \phi_B \equiv \phi$, $\Delta_A = \Delta_B \equiv \Delta$, $\hat{\sigma}_A = \hat{\sigma}_B \equiv \hat{\sigma}$, and $\sigma_A = \sigma_B = 0$. Suppose the economy is in steady state, and there is a one-time increase $\Delta_\tau - \Delta_{\tau-} > 0$ at time τ. If Δ_τ is large enough, and if condition (95) in Appendix C.5 holds, then an endogenous bust occurs.*

Intuitively, Proposition 3.2 says: if diversification improves enough, financiers will *deterministically* hit their leverage constraint in finite time. At that time, distressed investors will begin participating which triggers two effects: (i) a potential loss in output through "distress costs" that scale with participation; and (ii) a sudden increase in expected returns that rebuilds financier wealth and pushes them away from the leverage constraint. If the rebuilding effect (ii) is sufficiently weak, then the economy gets temporarily stuck at the leverage constraint, inducing non-negligible participation by distressed investors, hence non-negligible distress costs. Condition (95) ensures a weak rebuilding effect.

Next, we consider financial instability, which is also tightly associated with a binding leverage constraint.

Definition 3. *We say* financial instability *occurs if investment and consumption are more volatile than capital, i.e.,* $Var_t[dY_t] > 0$*, where* $Y := \iota + x\eta\rho_F + (1 - x\eta)\rho$.

Proposition 3.3. *Consider the equilibrium of Proposition C.3 with* $G_A = G_B \equiv G$*,* $\phi_A = \phi_B \equiv \phi$*, and* Δ_A, Δ_B *chosen large enough. Then, there is financial instability if and only if the leverage constraint (17) binds.*

Instability shows up in both sectors' risk premia. Figure 8 shows the IRFs of lending spreads after a gradual improvement in sector A diversification.[27] Once leverage constraints begin to bind, a large right tail appears suddenly in both sectors. Although the diversification improvement is sector-specific, the sudden possibility of extreme spreads emerges in both sectors, because distressed investors typically enter both sectors when financiers are leverage-constrained. This tail event, with spillovers to all sectors, resembles a financial crisis. Recent literature has documented low spreads leading into a predictable downturn or financial crisis,[28] which is a feature of these diversification-induced dynamics.

This finding also sheds light on the timing of the 2008 financial crisis. In the model, a possible sequence of events is as follows: (i) mortgage diversification improves (Δ_A increases exogenously); (ii) there is a reallocative housing boom (κ increases); (iii) financial leverage builds ($\lambda_F^A + \lambda_F^B$ increases); (iv) housing slows down (Z^A decreases exogenously); (v) financial crisis ($\lambda_F^A + \lambda_F^B = \bar{\lambda}$). Here, it is possible to have a housing slowdown in 2006, with a crisis and associated spillovers two years later.

[27] Lemmas 2.4 and 2.5 continue to hold in this economy, so these IRFs can be interpreted as responses to anticipated improvements rather than a series of zero-probability events.

[28] López-Salido et al. (2017) and Krishnamurthy and Muir (2016) show credit spreads tend to be low even though the bust is predictable (also see Schularick and Taylor (2012)). Baron and Xiong (2017) show bank equity provides low returns at the height of the boom, even though bank riskiness is elevated, measured by crash risk.

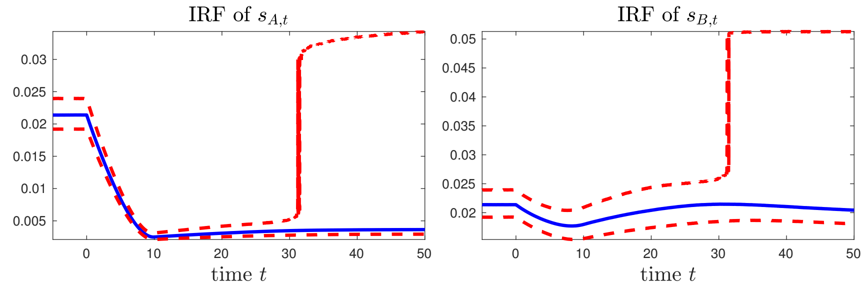

Figure 8: IRFs to a gradual increase from $\Delta_A = 0.5$ to $\Delta_A = 1$ from time $t = 0$ to $t = 10$. Solid lines are median responses, and dashed lines are 5th and 95th percentile responses. Parameters: $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $G_A = G_B = 0.1$, $\Delta_B = 0.5$, $\rho = 0.02$, $\rho_F = 0.06$, $\chi = 0.05$, and $\bar{\lambda} = 10$.

4 Quantification: US Housing Cycle

The objective of this section is to quantify the reallocation and leverage effects in the context of the 1990s-2000s US housing cycle. The first step is to determine a reasonable size for the diversification shock (Section 4.1). The second step is to calibrate the model to fit this particular episode (Section 4.2). The "test" is whether the model generates plausible dynamics for financier leverage and lending spreads, series not targeted by calibration. I show that the model without diversification improvements cannot qualitatively generate the same dynamics.

4.1 Measuring Diversification

In this section, I construct a quantitative measure of mortgage-market diversification. At a high level, the steps involved are as follows. First, I construct synthetic mortgage portfolios for mortgage lenders, using originations data in the HMDA dataset. For loans that are sold or securitized, I assume they are 100% diversified. Loans that are held on the lender's balance sheet are imperfectly diversified, and computing the exact degree of diversification follows the instructions below. The result is therefore a holistic measure of diversification, accounting for loan sales to Fannie/Freddie, securitizations, and geographic diversification.

Second, I compute the one-year-ahead volatility of each lender's mortgage portfolio, using location-specific house-price changes as the proxy for each loan's return.[29] The

[29]In doing this, I am assuming the risk on lender's mortgage portfolios can be proxied by the risk inherent in the house prices to which the mortgages are attached (or at least assuming the mortgage risk is proportional to the house-price risk). This proportionality assumption is incorrect per se, mainly because mortgages are debt contracts, which can be thought of as nonlinear functions of the local house price (e.g., default in bad states). But my assumption is reasonable as long as the covariances between the house prices in different locations are similar to the covariances between mortgage payments in different

lender's portfolio return is simply a weighted average of these loan-level returns, and I compute the volatility of this return. Importantly, this method automatically accounts for the empirical correlation between loans held on a lender's balance sheet. Denote the average lender-level volatility $\hat{\sigma}_{\Delta,t}$. Then, I proxy loan-level risk by measuring the average of all locations' one-year-ahead house-price volatility. Denote this average location-level volatility $\hat{\sigma}_t$. Finally, I back out time-varying diversification Δ_t using the model-implied relationship $(1-\Delta_t)\hat{\sigma}_t = \hat{\sigma}_{\Delta,t}$. Details on this procedure are in Appendix F.2.

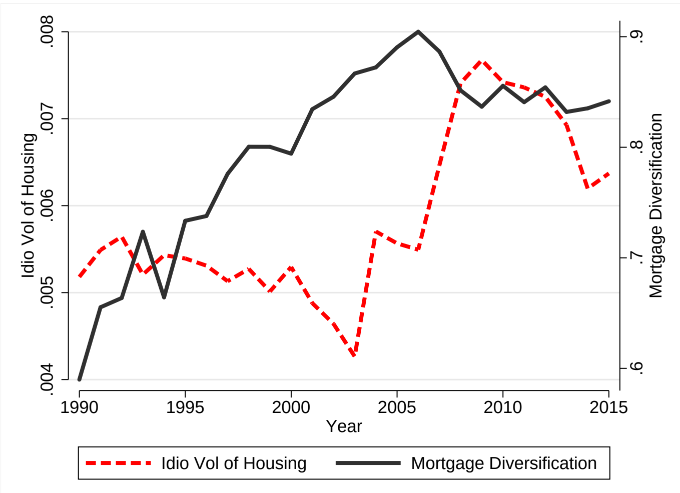

Figure 9: Diversification Index. "Idio Vol of Housing" plots estimates of $\hat{\sigma}_t$. "Mortgage Diversification" plots estimates of Δ_t. In this figure, the definition of "location" is a county. Source: HMDA and CoreLogic.

In figure 9, I plot the diversification index, Δ_t. In 1990, under 60% of housing risk was diversified by lenders.[30] By 2005, over 90% of such risk was diversified. During the same time period, the idiosyncratic volatility of housing ($\hat{\sigma}_t$) was not significantly reduced, indicating lenders faced lower housing risks primarily due to diversification.

Why did diversification increase so dramatically? I find both securitization and geographic diversification were significant factors. Figure 10 shows the number of counties represented by loans in an average lender's portfolio increased from 10 to 30 during the boom. During the same time, the fraction of mortgage loans sold (either to Fannie/Freddie or to private-label securitizations) increased from 45% to 60%. The geographic diversification seems to have been under-appreciated during this episode.

4.2 Calibrated Model

In this section, I interpret sector A as housing, and sector B as all other productive capital. The parameters and targets for this model are listed in table 2.

locations, because these covariances are the key inputs in how I measure diversification.

[30] I do not include the 1980s due to HMDA data limitations. As discussed in Mian et al. (2017b) and Fieldhouse et al. (2018), banking deregulations and mortgage securitizations (by GSEs) began aggressively

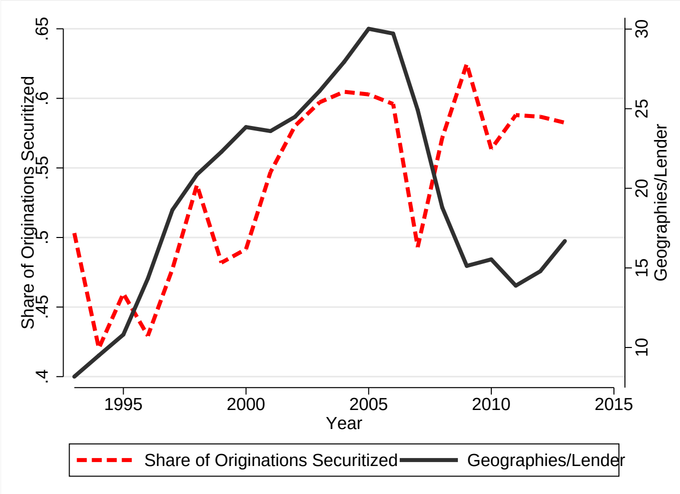

Figure 10: Diversification Components. "Share of Originations Securitized" shows the fraction of mortgage originations that are securitized within the same year of origination. "Geographies/Lender" computes the average number of counties per lender for loans held on lender's balance sheets. Source: HMDA.

Into the model, I feed in a series for $\Delta_{A,t}$ that approximately matches figure 9. I assume $\Delta_{A,t} = 0.59$ for $t \in [1980, 1990]$. Then, $\Delta_{A,t}$ increases linearly from 1990 until 2006, where $\Delta_{A,2006} = 0.91$. The resulting series is depicted in the top left panel of figure 11.

To extract the two-dimensional Brownian shocks (Z_t^A, Z_t^B), I approximately match two model-implied series to the data, from 1980 to 2015: (a) log GDP and (b) the log household credit share (each with 50% weights) i.e.,

$$\log(\text{GDP}) = \log\Big(K[\kappa G_A + (1-\kappa)G_B - (1-x)\eta(\lambda_D^A + \lambda_D^B)]\Big)$$
$$\log(\text{household credit share}) = \log\left(\frac{(\kappa+0.1)\phi_A}{(\kappa+0.1)\phi_A + (1-\kappa)\phi_B}\right).$$

The 0.1 wedge in the household credit share is to account for the fact that mortgage credit only accounts for approximately 2/3 of household credit. The extracted shock series are depicted in the top right panel of figure 11. In all figures, "model" refers to the model with shocks to both Δ_A and (Z^A, Z^B). "Counterfactual" refers to the model with the same shocks to (Z^A, Z^B) but assumes Δ_A constant. In both the model with diversification shocks and the counterfactual exercise, I use the binomial approximation to Brownian motion, i.e., $dZ = \pm\sqrt{dt}$. Both exercises are initialized with the state variables (α, η, x) at their stationary averages.

The data series used for shock extraction, and their model counterparts, are depicted in the bottom panels of figure 11. Both models roughly match aggregate output and household credit. However, we see the counterfactual model's shocks are larger, thus less likely from an ex-ante perspective. For example, the probability of observing $|Z^A - Z^B|$

in the 1980s.

	Parameter	Value	Targets	
Panel A: Fundamentals				
G_A	productivity	0.04	housing average return	0.04
G_B	productivity	0.06	capital-housing wealth ratio*	3
$\|\sigma_A\|$	aggregate vol	0.03	aggregate house price vol	0.03
$\|\sigma_B\|$	aggregate vol	0.053	output growth vol*	0.04
$\hat{\sigma}_A$	idiosyncratic vol	0.11	idiosyncratic house price vol	0.11
$\hat{\sigma}_B$	idiosyncratic vol	0.25	idiosyncratic stock price vol	0.25
Panel B: Preferences / OLG				
ρ	discount rate	0.02	riskless rate*	0.02
ρ_F	discount rate	0.06	output growth rate*	0.03
δ	birth/death rate	0.02	life expectancy	50
ν_F	population share	0.01	financier+distressed leverage*	5
ν_A	population share	0.09	housing consumption share*	0.22
ν_B	population share	0.85	aggregate Sharpe ratio*	0.20
ν_D	population share	0.05	$\nu_F + \nu_D + \nu_A + \nu_B = 1$	
Panel C: Financing				
ϕ_A	liability-asset ratio	0.4	aggregate housing LTV	0.4
ϕ_B	liability-asset ratio	0.26	household credit share*	0.42
Δ_A	diversification	0.59	1990 mortgage diversification	0.59
Δ_B	diversification	0.90	syndicated bank loan spread*	0.015
$\bar{\lambda}$	maximal leverage	14	binding constraint probability*	0.03
χ	distress cost	0.03	maximal funding cost increase*	$0.02 - 0.03$

Table 2: Parameter values and targets. Housing moments are taken from Piazzesi and Schneider (2016) and Davis and Van Nieuwerburgh (2015). Idiosyncratic stock volatility is from Di Tella (2017). Financial leverage and crisis probability (binding constraint probability) are from He and Krishnamurthy (2014). The syndicated loan spread is from Sufi (2007). The 0.42 household credit share is the 1985 value of the series in figure 1. In the model, household credit share is computed as $(\kappa + 0.1)\phi_A / ((\kappa + 0.1)\phi_A + (1-\kappa)\phi_B)$ to account for the approximately 1/3 share of household finance that was non-mortgage finance in the 1980s ($\kappa = 0.25$ with a capital-housing ratio of 3). The maximal funding cost increase is taken from the $> 2\%$ estimate of Fleckenstein and Longstaff (2018). Targets with stars (*) are only matched approximately.

increase by at least 22 over 10 years, as in the counterfactual from 2000-2010, is equal to 8.6×10^{-7}. By contrast, the model with diversification improvements implies $|Z^A - Z^B|$ increases by 12 over the same period, which has probability 7.3×10^{-3}, four orders larger.[31] The large counterfactual expansion of $|Z^A - Z^B|$ from 2000-2010 is needed to explain the large growth in household credit, which diversification improvements naturally generate through the reallocation effect. Repeating this analysis for 2005-2010 reveals probabilities of 0.0057 (counterfactual) and 0.1714 (model).

As shown in figure 12, the model with diversification also generates different distress

[31] These probabilities are calculated using the fact that $M := |Z^A - Z^B|/\sqrt{2}$ is a *reflected Brownian motion*. The probability distribution is given by $\mathbb{P}(M_t \geq m) = 2\Phi(-m/\sqrt{t})$, where $\Phi(\cdot)$ denotes the standard normal cdf.

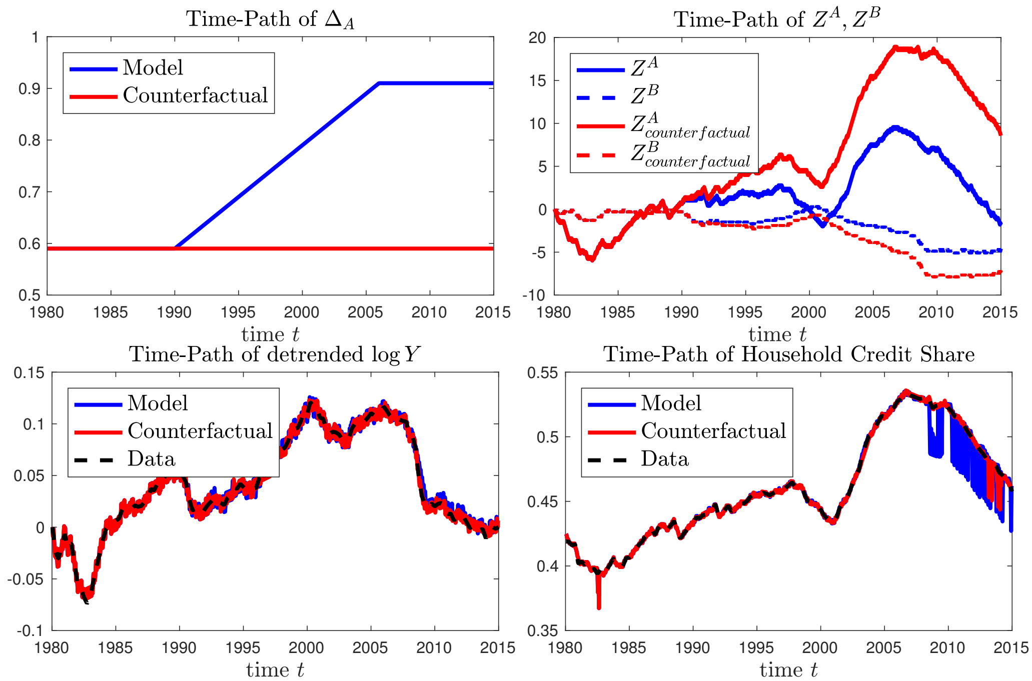

Figure 11: Shocks (top panels) and matched series (log GDP in left panel; household credit share in right panel). Parameters in table 2.

and financial market patterns. Without diversification shocks, financier leverage does not build up, and leverage constraints are no concern. Before the distress, during the boom years, diversification improvements reduce sector A spreads, as in the data. This force operates somewhat independently of sector B spreads. But as distress arises, s_A and s_B spike about 2.5% and move together thereafter, nearly one-for-one. Spreads move more closely together in busts, because their behavior is determined by financiers' health issues, rather than sectoral concerns. In this period, spreads reflect almost exactly the behavior of the shadow-funding cost ζ.

Quality Gradient. The previous calibration uses parameters (e.g., LTV ϕ_A and idiosyncratic risk $\hat{\sigma}_A$) relevant for the *average* household borrower. What are the effects of diversification improvements if the model is instead calibrated to *marginal* household borrowers, who tend to be riskier (higher $\hat{\sigma}_A$) and more external-finance dependent (higher ϕ_A)?

This question is interesting for two reasons. First, diversification improvements in the 1990s and especially 2000s were likely larger for lower-quality borrowers. For example, securitization of non-conforming loans (private-label MBS containing subprime, alt-A, and jumbo loans) increased dramatically in the 2000s, even relative to conforming loans (see Appendix F.1). By contrast, figure 9 only shows the average increase in

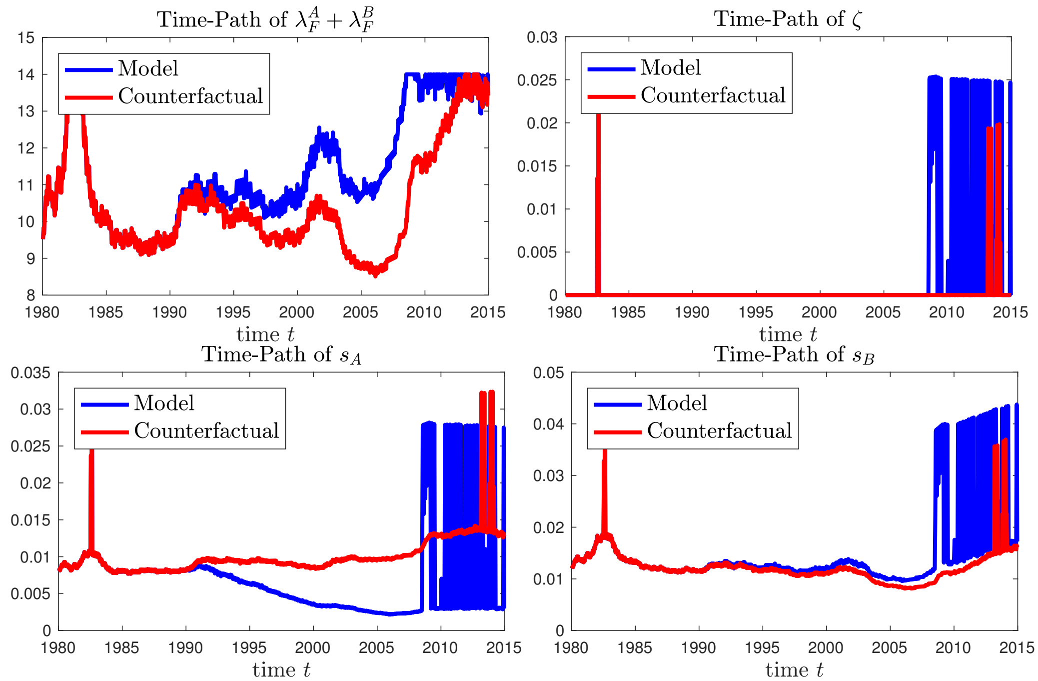

Figure 12: Financier leverage (top left panel), shadow funding cost (top right panel), and lending spreads (bottom panels). Parameters in table 2.

diversification over all mortgage originations, which hides any potential heterogeneity.

Second, for a given increase in diversification, the effects are likely to be stronger. Theoretically, suppose sector A has a greater amount of idiosyncratic risk ($\hat{\sigma}_A > \hat{\sigma}_B$) and borrows against a greater fraction of its asset purchases ($\phi_A > \phi_B$).[32] Then, a diversification boom in sector A tends to produce larger reallocation and leverage effects than a diversification boom in sector B. The intuition comes from idiosyncratic risk prices: larger $\hat{\sigma}_A$ or ϕ_A imply risk price reductions from diversification, i.e., $\frac{d^2\hat{\pi}_{F\to A}}{d\Delta_A d\hat{\sigma}_A} < 0$ and $\frac{d^2\hat{\pi}_{F\to A}}{d\Delta_A d\phi_A} < 0$, holding fixed κ. Thus, improved diversification of lower-quality borrowers' risks might start a large cycle, and it helps reconcile the timing of the housing boom with the 2000s private-label MBS boom, rather than an earlier increase in diversification of conforming mortgages.[33]

I increase ϕ_A from 0.40 to 0.60 and $\hat{\sigma}_A$ from 0.11 to 0.20. To match the targets in table 2, I also adjust the following parameters: $G_A = 0.05$; $\rho_F = 0.155$; $\nu_A = 0.50$; $\nu_B = 0.44$; $\phi_B = 0.389$; and $\bar{\lambda} = 27$.[34] The procedure for extracting (Z^A, Z^B) is the same as before.

Figure 13 shows financier leverage, financiers' shadow-funding cost, and both sectoral spreads. Under this calibration, the boom coincides with a sustained decline in

[32] If one interprets ϕ_z as borrowing demand, it is natural to assume poorer insiders will have higher ϕ_z. Alternatively, lower-quality insiders might have higher issuance under asymmetric information about insider types. In a standard signalling equilibrium, higher-quality types must retain a greater share of risk

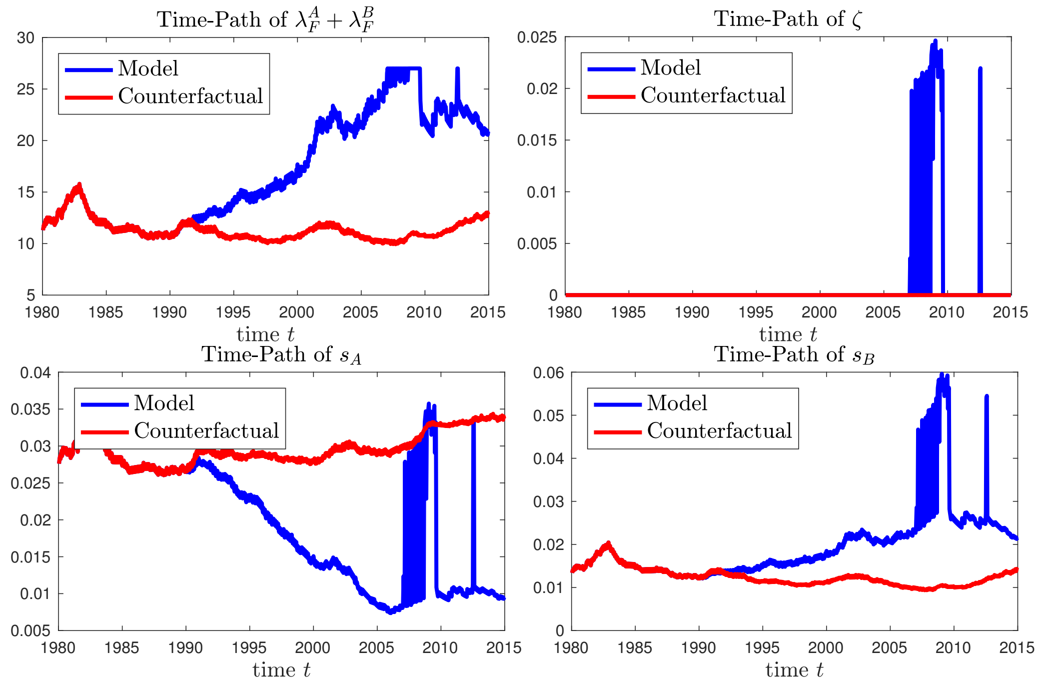

Figure 13: Financier leverage (top left panel), shadow funding cost (top right panel), and lending spreads (bottom panels). Parameters in table 2, with exception of the modifications in the text.

s_A, on the order of 2%, in line with the drop in spreads documented in Justiniano et al. (2017), and twice as large as the baseline parameterization. At the same time, diversification-induced leveraging is massive: financiers lever up from 12 to 27 between 1990 and 2007, which is a similar order of magnitude to the broker-dealer leverage increase documented in figure 1. These strengthened effects can be attributed to the discussion above: diversification improvements have a larger impact on riskier (higher $\hat{\sigma}_A$) and more external-finance-dependent (higher ϕ_A) borrowers.

In 2007, leverage constraints are hit, and a financial crisis occurs, upon which spreads in both sectors jump by about $\chi = 3\%$, remaining elevated for longer than in the baseline calibration. The counterfactual without diversification improvements generates no financial crisis at all.

in order to separate themselves from low-quality types.

[33] Mian and Sufi (2018) argue the private-label MBS boom caused the housing boom.

[34] This calibration, mainly due to the high value of $\bar{\lambda}$, reflects a 0.1% probability of leverage constraints binding, under $\Delta_A = 0.59$. This is significantly lower than the 3% target from table 2, but this target is intentionally underestimated, because diversification improvements are more consequential under this calibration, as figure 13 illustrates.

5 Conclusion

A sector-specific diversification improvement is a credit supply shock that can generate a sectoral boom followed by an economy-wide bust. During the boom, spreads are particularly low, but can eventually spike as the economy enters into a predictable financial crisis, sprung by slowly-building financial leverage. The recent US housing boom, and the ensuing 2008 financial crisis, appears to be a good example. Unlike many models that focus in on borrower distress, my narrative is solely concerned with the risk-reward trade-offs of intermediaries.

My quantitative application focuses on the recent housing boom, but as Mian et al. (2017a) show, household credit predicts future recessions systematically better than non-household credit. What is special about housing, as it pertains to boom-bust cycles? Future work should go beyond exogenous housing-specific shocks and try to understand why the effects of neutral-seeming financial shocks (such as a global savings glut) might be stronger in housing markets.

Finally, this paper builds off an exogenous shift in diversification, but such financial innovations are endogenous. Studying this innovation process – e.g., marketing of securitized products, creation of robust banking networks, and even financial deregulations – could uncover rich linkages to other financial variables, such as credit standards and collateral constraints.

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Appendix – for online publication

A Results on the Brownian Cylinder W

A.1 Aggregate Risk Along Investment Arcs

Proof of Lemma 2.2. To examine the degree of aggregate risk in this economy, consider investing one unit of consumption, divided equally amongst each market in $[\frac{1-\Delta}{2}, \frac{1+\Delta}{2}]$ (the fact that it is centered at $1/2$ is without loss of generality, by symmetry). This results in:

$$\begin{aligned}
\mathrm{Var}_t\Big(\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} \Delta^{-1} dW_{i,t} di\Big) &= \Delta^{-2}\mathrm{Cov}_t\Big(\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} dW_{i,t} di, \int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} dW_{j,t} dj\Big) \\
&= \Delta^{-2}\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}}\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} \mathrm{Cov}_t(dW_{i,t}, dW_{j,t}) di dj \\
&= \Big(1-6\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}}\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} \Delta^{-2}|i-j|(1-|i-j|) di dj\Big) dt \\
&= \Big(1-6\int_0^1\int_0^1 \Delta|x-y|(1-\Delta|x-y|) dx dy\Big) dt \\
&= \Big(1-6\int_{-1}^1 (1-|u|)(1-\Delta|u|)\Delta|u| du\Big) dt \\
&= (1-\Delta)^2 dt.
\end{aligned}$$

In the third line, I have substituted the covariance and distance metric: $\mathrm{Cov}_t(dW_{i,t}, dW_{j,t}) = 1 - 6\min(|i-j|, 1-|i-j|)(1-\min(|i-j|(1-|i-j|)) = 1-6|i-j|(1-|i-j|)$. In the fourth line, I have performed the change-of-variables $i = \frac{1-\Delta}{2} + \Delta x$ and $j = \frac{1-\Delta}{2} + \Delta y$. In the fifth line, I have substituted $u = x - y$ and used the fact that if X and Y are independent uniform random variables, then $X - Y$ has the triangular distribution. Given this formula, we may take $\Delta \to 1$ to see that $\mathrm{Var}_t(\int_0^1 dW_{i,t} di) = 0$. As this expectation is zero, this shows that $\int_0^1 dW_{i,t} di = 0$ almost-surely. □

A.2 Existence of W

One may ask whether or not such a stochastic process $W := \{W_{i,t} : i \in [0,1], t \geq 0\}$ exists on any probability space. In other words, are the properties assumed above mutually consistent? Below, I prove that such shocks exist by an implicit method, using the theory of Gaussian processes.

This relates to the class of Gaussian random fields that are used to model forward rates in Kennedy (1994), which to my knowledge is the first use of such processes in financial economics. The key property aiding the analysis of that paper, as in this paper, is the independent increments property of the random field in the "time" direction. Santa-Clara and Sornette (2001) study a similar stochastic process, which they call "string" shocks. They obtain these shocks using the theory of stochastic partial differential equations (SPDEs), although I prove existence in a different way. That said, the W process is not a special case of the class of processes they consider. Furthermore, my existence proof is general enough to apply analogously to their entire class of processes.

First, I build a particular Gaussian process. Second, I show that this stochastic process has the desired properties. Given the construction, which posits the covariance in the t-direction and i-direction as multiplicatively separable, and the property that the process acts as a continuum of Wiener processes in the t-direction, W is thus an example of a cylindrical Wiener process (see a reference on SDEs in infinite dimensions, e.g., Da Prato and Zabczyk (2014)).

Proof of Lemma 2.1. The existence of a mean-zero Gaussian process having covariance function

$$V((i,s),(j,t)) = \Big[1 - 6\mathrm{dist}(i,j)(1-\mathrm{dist}(i,j))\Big] \times \min(s,t)$$

is guaranteed if and only if V is symmetric and positive semi-definite (see any reference on Gaussian processes, e.g., proposition I.24.2 in Rogers and Williams (2000)). Clearly, V is symmetric. To check positive semi-definiteness, construct the Gram matrix: let $i_1, \ldots, i_N \in [0,1]$ and $t_1, \ldots, t_N \in \mathbb{R}_+$, and define the matrix G by

$$G := [V((i_m, t_m), (i_n, t_n))]_{m,n\in\{1,\ldots,N\}}.$$

We need to show that G is positive semi-definite. To do this, define the "univariate" covariance functions $v_1(i,j) := V((i,1),(j,1))$ and $v_2(s,t) := V((0,s),(0,t))$, and the associated Gram matrices

$$G_1 := [v_1(i_m, i_n)]_{m,n\in\{1,\ldots,N\}} \quad \text{and} \quad G_2 := [v_2(t_m, t_n)]_{m,n\in\{1,\ldots,N\}}.$$

Notice that

$$G = G_1 \circ G_2,$$

where $\circ$ denotes the Schur product (element-wise multiplication). By the Schur product theorem, it suffices to show that G_1 and G_2 are both positive semi-definite, because then so is G.

Consider a standard Brownian bridge process $\{W_i^\circ : i \in [0,1]\}$ and define the process

$$B_i := \sqrt{12}\Big[W_i^\circ - \int_0^1 W_j^\circ dj\Big].$$

Note that $\mathbb{E}B_i = 0$ for all i and

$$\begin{aligned}
\mathbb{E}B_iB_j &= 12\mathbb{E}[(W_i^\circ - \int_0^1 W_k^\circ dk)(W_j^\circ - \int_0^1 W_k^\circ dk)] \\
&= 12\Big[\mathbb{E}W_i^\circ W_j^\circ + \mathbb{E}\int_0^1\int_0^1 W_k^\circ W_l^\circ dkdl - \mathbb{E}\int_0^1 W_i^\circ W_k^\circ dk - \mathbb{E}\int_0^1 W_j^\circ W_k^\circ dk\Big] \\
&= 12\Big[\min(i,j) - ij + \int_0^1\int_0^1 [\min(k,l) - kl]dkdl - \int_0^1 [\min(i,k) - ik]dk - \int_0^1 [\min(j,k) - jk]dk\Big] \\
&= 12\Big[\min(i,j) - ij + \int_0^1 \frac{l(1-l)}{2}dl - \frac{i(1-i)}{2} - \frac{j(1-j)}{2}\Big] \\
&= 1 - 6|i-j|(1-|i-j|).
\end{aligned}$$

In the third and fourth equality, I have used the Brownian bridge covariance function to compute $\mathbb{E}W_i^\circ W_j^\circ = \min(i,j) - ij$, as well as the integral

$$\begin{aligned}
\int_0^1 [\min(i,j) - ij]dj &= \int_0^i j(1-i)dj + \int_i^1 i(1-j)dj \\
&= \frac{1}{2}i^2(1-i) + \frac{1}{2}i(1-i)^2 \\
&= \frac{i(1-i)}{2}.
\end{aligned} \tag{22}$$

Therefore, v_1 is the covariance function for B. As a valid covariance function, we immediately conclude that G_1 is positive semi-definite. Finally, v_2 is the covariance function of standard Brownian motion, so the matrix G_2 is positive semi-definite.

Thus, define W to be a Gaussian process with covariance function V. We want to show that W has the desired properties from Assumption 1 of the text: (1) at each location, W acts as a Brownian motion; (2) dW has the correct cross-sectional correlations; (3) W has a path-continuous version.

First, fixing i, the time-series process $W^{(i)} := \{W_{i,t} : t \geq 0\}$ is a standard Brownian motion. Indeed, $\mathbb{E}[W_{i,0}^2] = 0$ implies $W_{i,0} = 0$ almost-surely. Since $W^{(i)}$ is a centered Gaussian process with $V((i,s),(i,t)) = \min(s,t)$, it has the same probability law as a standard Brownian motion. Having the same probability law, it is well-known that $W^{(i)}$ can be chosen to be path-continuous. Independent increments can be established as follows. Using the covariance function and the Normal distribution, we have $\mathbb{E}[(W_{i,t_2} - W_{i,t_1})(W_{i,t_1} - W_{i,t_0})] = 0$ for $t_2 \geq t_1 \geq t_0 \geq 0$. Orthogonality plus joint Normality implies independence of $W_{i,t_2} - W_{i,t_1}$ from $W_{i,t_1} - W_{i,t_0}$.

Second, the increments to $W^{(i)}$ and $W^{(j)}$ have the desired pairwise correlations. Indeed, using the covariance function V, we have

$$\frac{1}{s}\mathbb{E}[(W_{i,t+s} - W_{i,t})(W_{j,t+s} - W_{j,t})] = 1 - 6\text{dist}(i,j)(1 - \text{dist}(i,j)).$$

As $s > 0$ is arbitrary, and using the Markov property of Brownian motion, we have that

$$\text{corr}(dW_{i,t}, dW_{j,t} \mid \mathcal{F}_t) = 1 - 6\text{dist}(i,j)(1 - \text{dist}(i,j)).$$

Third, we can use the Kolmogorov-Chentsov continuity criterion (see any reference on Gaussian processes, e.g., theorem I.25.2 in Rogers and Williams (2000)) to show that W has a version with continuous sample paths.[35] To do this, we may fix an arbitrary $T > 0$ and show that there exist $C > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$ (which may all depend on T) such that

$$\mathbb{E}|W_{i,s} - W_{j,t}|^{\varepsilon_1} \leq C \times \text{dist}((i,s),(j,t))^{2(1+\varepsilon_2)}, \quad \forall s,t \leq T, \tag{23}$$

where $\text{dist}(\cdot,\cdot)$ is Euclidean distance in $\mathbb{C}_1^\circ \times \mathbb{R}$, where $\mathbb{C}_1^\circ$ is the circle of circumference one.[36] In particular,

$$\text{dist}((i,s),(j,t)) := \sqrt{|s-t|^2 + |i-j|^2(1-|i-j|)^2}.$$

Assume $s > t$ (the opposite case follows symmetrically). Set $\varepsilon_2 > 0$ arbitrarily, and set $\varepsilon_1 = 4(1+\varepsilon_2)$. Then, because W is Gaussian, there exists a constant M such that

$$\begin{aligned}\mathbb{E}|W_{i,s} - W_{j,t}|^{\varepsilon_1} &= \mathbb{E}|W_{i,s} - W_{j,t}|^{4(1+\varepsilon_2)} \\ &= M\mathbb{E}[|W_{i,s} - W_{j,t}|^2]^{2(1+\varepsilon_2)}.\end{aligned}$$

Compute, using the triangle inequality, the covariance function, and the assumption that $t < s < T$:

$$\begin{aligned}\mathbb{E}|W_{i,s} - W_{j,t}|^2 &= V((i,s),(i,s)) + V((j,t),(j,t)) - 2V((i,s),(j,t)) \\ &\leq |V((i,s),(i,s)) - V((i,s),(j,t))| + |V((j,t),(j,t)) - V((i,s),(j,t))| \\ &\leq 2|V((i,s),(j,t)) - V((j,t),(j,t))| + |s-t| \\ &= 2|\min(s,t)(1 - 6|i-j|(1-|i-j|)) - t| + |s-t| \\ &= 12t|i-j|(1-|i-j|) + |s-t| \\ &\leq 24\max(T, 1/12)[|s-t|^2 + |i-j|^2(1-|i-j|)^2]^{1/2} \\ &= 24\max(T,1/12)\text{dist}((i,s),(j,t))\end{aligned}$$

Consequently,

$$\mathbb{E}|W_{i,s} - W_{j,t}|^{\varepsilon_1} \leq M(24\max(T,1/12))^{2(1+\varepsilon_2)}\text{dist}((i,s),(j,t))^{2(1+\varepsilon_2)},$$

which is condition (23) with $C = M(24\max(T,1/12))^{2(1+\varepsilon_2)}$. □

[35] One might think we could prove continuity by appealing to the fact that $\{W_{i,t} : i \in [0,1]\}$ is a translated, scaled Brownian bridge for each t, and $\{W_{i,t} : t \geq 0\}$ is a Brownian motion for each i. Thus, we could construct continuous versions of each of these at the rational indexes, and use a density argument to construct a continuous W in the limit. The problem with this approach is that we don't know that the limiting process has the desired distributional properties.

[36] This is a slight generalization of the conventional Kolmogorov-Chentsov theorem, in which the index set is not $\mathbb{R}^2$. But the exact same condition applies.

B Micro-foundations and Optimization Problems

B.1 Micro-foundation of Skin-in-the-Game Constraint

In the model of Section 2, an insider sells an exogenous fraction ϕ of the capital stock to outsiders (financiers). The insider keeps a fraction of $1-\phi$ of the capital risk on his own balance sheet. In this appendix, I derive this risk-sharing arrangement as the approximate solution to a standard moral hazard problem.

Let the capital stock of a generic insider evolve as follows:

$$dk_{i,t} = k_{i,t}[(\iota_{i,t} - \delta_{i,t})dt + \sigma dZ_t + \hat{\sigma} dW_{i,t}].$$

The new object is $\delta_{i,t}$, which captures hidden diversion. In particular, insiders may divert $\delta k dt$ units of capital to obtain $(1-\phi)\delta k dt$, where ϕ determines the inefficiency from diversion. This may also be thought of as diverting effort away from capital upkeep.

Insiders hold assets, borrow/lend in risk-free debt markets, and make contractual payments to outsiders. Contractual payments are $-k_{i,t} d\Omega_{i,t}$ per unit of time, since the price of capital is unity in the absence of investment adjustment costs. Since diversion is unobservable, Ω_i must be adapted to the principal's information set, which is generated by (Z, W_i^δ), where $dW_{i,t}^\delta := dW_{i,t} - \delta_{i,t}dt$ is the ex-diversion shock. Contract payments thus take the form $d\Omega_{i,t} = \zeta_{i,t}[(\varpi_{i,t} - \delta_{i,t})dt + \hat{\sigma} dW_{i,t}] + \gamma_{i,t} \cdot dZ_t$ for some processes $(\zeta_{i,t}, \varpi_{i,t}, \gamma_{i,t})$ adapted to (Z, W_i^δ).[37] If both insiders and outsiders may frictionlessly trade claims on the aggregate shock Z, the choice of γ_i is irrelevant. When we assume insiders may not trade claims on Z, i.e., $\theta \equiv 0$, then we are implicitly restricting $\gamma_i = \zeta_i \sigma$. Thus, we use this latter assumption, as it is without loss of generality in the former case. Incorporating these contract payments, insiders' net worth evolution is

$$\begin{aligned} dn_{i,t} = \; & \underbrace{(n_{i,t} r_t - c_{i,t})dt}_{\text{consumption-savings}} + \underbrace{(1-\phi)\delta_{i,t} k_{i,t} dt}_{\text{diversion benefits}} + \underbrace{k_{i,t}(dR_{i,t} - r_t dt)}_{\text{excess return-on-assets}} \\ & - \underbrace{k_{i,t}\zeta_{i,t}\Big[(\varpi_{i,t} - \delta_{i,t})dt + \hat{\sigma} dW_{i,t} + \sigma \cdot dZ_t\Big]}_{\text{contract payments}} + \underbrace{n_{i,t}\theta_{i,t} \cdot (\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}. \end{aligned}$$

This budget constraint has a simple interpretation. Insiders retain a stake $1-\zeta$ in their asset risks and issue ζ to outsiders. This can be thought of as an equity stake, which has expected excess return ϖ.

Definition 4. Optimal contracts *consist of possible risk exposures and promised payments (i.e., $\zeta_{i,t}$, $\varpi_{i,t}$) that implement no diversion (i.e., $\delta_{i,t} \equiv 0$ for all i, t) and maximize total surplus in the following sense. Taking as given future contracts $\{\zeta_{i,t+s}, \varpi_{i,t+s}\}_{s>0}$, time-$t$ contracts $(\zeta_{i,t}, \varpi_{i,t})$ maximize total instantaneous surplus among contracting parties.*

An important feature of these contracts is that they are short-term, which is captured by the last statement in Definition 4. Contracts are chosen to maximize instantaneous surplus, rather than total long-term surplus, which aids tractability. These short-term contracts would be optimal long-term contracts as well, if agents cannot commit to future contracts and if those future contracts are made in anonymity.

To derive optimal contracts, note that diversion of δk units of capital yields $(1-\phi)\delta k$ in net worth to the insider. On the other hand, the insiders' return-on-assets is reduced by δ, which translates into a $(1-\zeta)\delta k$ lower payoff from inside equity. Consequently, the insider will not divert any capital as long as $\zeta_{i,t} \leq \phi$, which is insiders' incentive-compatibility constraint. In other words, $1-\phi$ is the minimum skin-in-the-game requirement.

Given competition in the financier sector, ϖ is determined by their marginal utility process, i.e., $\varpi = \hat{\sigma}\hat{\pi} + \sigma \cdot \pi$, where $\hat{\pi}$ is an idiosyncratic risk price (in equilibrium, $\hat{\pi} = (1-\Delta)\hat{\pi}_F$). The important thing about $\hat{\pi}$ is that it is independent of the ζ from this particular contracting problem. We can now write the return on inside equity without diversion,

$$dR_{i,t}^I := r_t dt + \frac{\mu_t - r_t - \zeta_{i,t}\hat{\sigma}\hat{\pi}_{i,t} - \zeta_{i,t}\sigma \cdot \pi_t}{1 - \zeta_{i,t}} dt + \sigma \cdot dZ_t + \hat{\sigma} dW_{i,t},$$

[37] Adaptability to W_i^δ implies the weights of $d\Omega_{i,t}$ on $\hat{\sigma} dW_{i,t}$ and $-\delta_{i,t}dt$ must be identical. This weight is $\zeta_{i,t}\hat{\sigma}$. The additional term $\zeta_{i,t}\varpi_{i,t}dt$ allows for time-varying flow payments.

where μ_t is the expected return-on-capital. Insiders' net worth can be re-written in terms of its inside equity position $e_{i,t} := (1-\zeta_{i,t})k_{i,t}$ and the return dR^I as

$$dn_{i,t} = (n_{i,t}r_t - c_{i,t})dt + e_{i,t}(dR^I_{i,t} - r_t dt) + n_{i,t}\theta_{i,t} \cdot (\pi_t dt + dZ_t).$$

The following facts simplify the analysis: (1) insiders can control their exposure $e_{i,t}$; and (2) financiers' surplus is accounted for by $\varpi_{i,t}$. Consequently, optimal $\zeta_{i,t}$ may be chosen by insiders as they wish, subject to the incentive constraint $\zeta_{i,t} \leq \phi$. Although I assume $\zeta_{i,t} = \phi$ in the main text, I detail the actual solution below which helps understand how good the assumption $\zeta_{i,t} = \phi$ is.

Since $\zeta_{i,t}$ only affects $\mathbb{E}_t[dR^I_{i,t}]$ and not $dR^I_{i,t} - \mathbb{E}_t[dR^I_{i,t}]$, optimal $\zeta_{i,t}$ is chosen to maximize $\mathbb{E}_t[dR^I_{i,t}] - r_t dt$, i.e.,

$$\max_{\zeta \in [0,\phi]} \Big\{ \frac{\mu - r - \zeta\hat{\sigma}\hat{\pi} - \zeta\sigma\cdot\pi}{1-\zeta} \Big\}.$$

It is optimal to set $\zeta = \phi$ when $\mu - r - \hat{\sigma}\hat{\pi} - \sigma\cdot\pi > 0$ and to otherwise set ζ such that $\mu - r - \hat{\sigma}\hat{\pi} - \sigma\cdot\pi = 0$. Doing this requires knowledge of the equilibrium risk prices, which depends on the market structure (e.g., whether or not θ is constrained or unconstrained). As an example, use the expressions from Proposition 2.3 for sector A, with ζ in place of ϕ, to get

$$\mu - r - \hat{\sigma}\hat{\pi} - \sigma\cdot\pi = (1-\zeta)\kappa\hat{\sigma}^2\Big[\frac{1-\zeta}{(1-\eta)\alpha} - \frac{\zeta(1-\Delta)^2}{\eta}\Big].$$

The solution is

$$\zeta_{i,t} = \min\Big(\phi,\ \Big[1 + (1-\Delta)^2\frac{\alpha_t(1-\eta_t)}{\eta_t}\Big]^{-1}\Big). \tag{24}$$

Notice that $\zeta_{i,t} = \phi$ is optimal across the state space if $\Delta = 1$, which is the case commonly studied in the literature (e.g., Di Tella (2017)). When financiers are imperfectly diversified, $\zeta_{i,t} < \phi$ is possible for very low values of η_t, because financiers' required rate of return diverges to infinity. However, in the steady-state equilibrium, wealth shares (55)-(56) are such that maximal issuance $\zeta_{i,t} = \phi$ is optimal. Furthermore, as Δ increases, the possibility of unconstrained risk-sharing shrinks. Notice that $\zeta_{i,t} < \phi$ when $\eta_t < \eta_t^*$, where

$$\eta_t^* := \frac{\phi\alpha_t(1-\Delta)^2}{1-\phi+\phi\alpha_t(1-\Delta)^2},$$

which shrinks to 0 at a quadratic rate as $\Delta \to 1$, i.e.,

$$\frac{d\log\eta^*}{d\log(1-\Delta)} = \frac{2[1-\phi+\phi\alpha(1-\Delta)^2 - \phi\alpha(1-\Delta)^2]}{1-\phi+\phi\alpha(1-\Delta)^2} = 2(1-\eta^*).$$

Hence, for relatively high values of Δ such as those considered in the quantitative section, the assumption of $\zeta_{i,t} = \phi$ is innocuous.

B.2 Generalization with Endogenous Credit Standards

By smoothing out the moral hazard problem of Appendix B.1, we can address the question of how diversification affects credit standards. To make these concepts precise, consider a more general hidden diversion technology. Suppose diverting $\delta k dt$ units of capital yields an income flow of $h(\delta)kdt$ to the insider, where $h(\cdot)$ satisfies the following.

Assumption 3 (Diversion Benefit). *Assume the function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is twice-differentiable with the following properties: $h(0) = 0$, $h'(\delta) \leq 1$ for all δ, $h''(\delta) \leq 0$ for all δ, and $h'(+\infty) = 0$.*

In Appendix B.1, we had assumed a linear diversion technology $h(\delta) = (1-\phi)\delta$, which satisfies Assumption 3. In this more general formulation, insiders' net worth evolves as

$$\begin{aligned} dn_{i,t} = {} & \underbrace{(n_{i,t}r_t - c_{i,t})}_{\text{consumption-savings}}dt + \underbrace{h(\delta_{i,t})k_{i,t}}_{\text{diversion benefits}}dt + \underbrace{k_{i,t}(dR_{i,t} - r_t dt)}_{\text{excess return-on-assets}} \\ & \underbrace{- k_{i,t}\zeta_{i,t}\Big[(\varpi_{i,t} - \delta_{i,t})dt + \hat{\sigma}dW_{i,t} + \sigma\cdot dZ_t\Big]}_{\text{contract payments}} + \underbrace{n_{i,t}\theta_{i,t}\cdot(\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}. \end{aligned}$$

As before, $(\zeta_{i,t}, \varpi_{i,t})$ characterize contract payments.

With this general specification of h, it may not be desirable to implement zero diversion. For instance, if $h'(0) = 1$, implementing no diversion requires insiders to keep 100% skin-in-the-game, i.e., $\zeta_{i,t} \equiv 0$. But such a contract may impose too much risk onto insiders' balance sheets. Thus, "optimal contracts" in this setting remove from Definition 4 the requirement $\delta_{i,t} \equiv 0$.

To solve this contracting problem, we may repeat a similar analysis to Appendix B.1. Given a skin-in-the-game ζ, optimal diversion maximizes $h(\delta) - (1-\zeta)\delta$. Thus, $h'(\delta) \leq 1-\zeta$. Since h' is weakly decreasing, optimal diversion is a weakly increasing function $f(\zeta)$. Next, optimal payments are $\varpi = f(\zeta) + \hat{\sigma}\hat{\pi} + \sigma \cdot \pi$, which now compensates financiers for the possibility that $\delta > 0$. As before, this means that it suffices to consider insiders' surplus, which is governed solely by their expected return on inside equity, i.e.,

$$\max_{\zeta \in [0,1]} \Big\{ \frac{\mu - r - f(\zeta) + h(f(\zeta)) - \zeta\hat{\sigma}\hat{\pi} - \zeta\sigma \cdot \pi}{1-\zeta} \Big\}.$$

Supposing f is differentiable and that optimal diversion is positive, $\delta > 0$, we have the first-order optimality condition

$$\mu - r - \hat{\sigma}\hat{\pi} - \sigma \cdot \pi = \zeta(1-\zeta)f'(\zeta) + f(\zeta) - h(f(\zeta)).$$

Modify the equilibrium expressions from Proposition 2.3 for sector A by putting ζ in place of ϕ and accounting for diversion benefits and costs:

$$\mu - r - f(\zeta) + h(f(\zeta)) - \sigma \cdot \pi = (1-\zeta)\hat{\sigma}\frac{\kappa(1-\zeta)\hat{\sigma}}{(1-\eta)\alpha} + \zeta(1-\Delta)\hat{\sigma}\frac{\kappa\zeta(1-\Delta)\hat{\sigma}}{\eta}.$$

Substitute the optimality condition for ζ to get the equilibrium condition:

$$\underbrace{(1-\zeta_{i,t})\zeta_{i,t}f'(\zeta_{i,t})}_{\text{marginal cost of issuance}} = \underbrace{\kappa_t(1-\zeta_{i,t})\hat{\sigma}^2\Big[\frac{1-\zeta_{i,t}}{(1-\eta_t)\alpha_t} - \frac{\zeta_{i,t}(1-\Delta)^2}{\eta_t}\Big]}_{\text{marginal benefit of issuance}}. \tag{25}$$

After dividing both sides by $1-\zeta$, the left-hand-side of (25) is strictly increasing in ζ, whereas the right-hand-side is strictly decreasing. Thus, there is a uniquely optimal skin-in-the-game in equilibrium. This equilibrium equates the "marginal benefit of issuance," comprised by diversification benefits from offloading risk, to the "marginal cost of issuance," defined by the marginal increase in diversion, net of the marginal private diversion benefits.[38]

With this smooth moral hazard setup, we can analyze the effects of diversification. From (25), higher Δ reduces optimal skin-in-the-game $1-\zeta$, which is a generalization of the results in Appendix B.1. In other words, insiders issue more securities to better-diversified outsiders. But interestingly, this now comes with a cost. Higher ζ increases equilibrium diversion $f(\zeta)$ and thus deadweight losses $f(\zeta) - h(f(\zeta))$. This result is analogous to the story that "securitization leads to lax screening" as in Keys et al. (2010).

B.3 Micro-foundation of Leverage Constraint

In Section 3, I have assumed the leverage constraint (17) is defined an exogenous and constant maximum $\bar{\lambda}$. Several macro-finance papers derive a bank leverage constraint from an underlying agency friction, e.g., the simple limited commitment problem in Gertler and Kiyotaki (2010). In that case, the constraint is not a constant level, but rather a function of the economic state and model parameters.

What is particularly important for my model is how diversification could affect the leverage constraint. Does better diversification relax the leverage constraint such that it would never bind in equilibrium? In this section, I use a simple agency problem similar to Gertler and Kiyotaki (2010) to derive the leverage constraint and show that it is insensitive to diversification improvements. Thus, the constraint is indeed more likely to bind as diversification improves.

We consider the model of Section 3 with symmetric discount rates (ρ), no OLG ($\delta = 0$), and a single productive sector ($G_A = G, G_B = -\infty$ such that $\kappa = 1$; I drop all A subscripts accordingly). Suppose financiers can abscond

[38]That is, the marginal cost is $(1-\zeta)\frac{d}{d\zeta}[f(\zeta) - h(f(\zeta))] = (1-\zeta)[f'(\zeta) - (1-\zeta)f'(\zeta)] = \zeta(1-\zeta)f'(\zeta)$.

with a fraction $\gamma \in (0,1)$ of their assets and renege on repayment of their short-term bonds. After doing this diversion, financiers would have net worth $\tilde{n}^F_{i,t} := \gamma \lambda_t n^F_{i,t}$. I consider two cases. After diversion, financiers move to another location and either (i) set up a new financial intermediary anonymously; or (ii) retire to become insiders in the single productive sector A. A fact I will use in both cases is the following: one can show that the value functions of financiers and insiders are given by $\log(n^F_{i,t}) + \xi_{F,t}$ and $\log(n^I_{i,t}) + \xi_{I,t}$, respectively.

In case (i), diversion delivers utility $\log(\tilde{n}^F_{i,t}) + \xi_{F,t}$, which must be ruled out by the incentive constraint $\log(\tilde{n}^F_{i,t}) + \xi_{F,t} \leq \log(n^F_{i,t}) + \xi_{F,t}$. As a result,

$$\lambda_t \leq \gamma^{-1}$$

is required. Therefore, the leverage limit $\bar{\lambda} \equiv \gamma^{-1}$ is completely independent of diversification.

In case (ii), the same analysis delivers the incentive constraint

$$\lambda_t \leq \gamma^{-1} \exp(\xi_{F,t} - \xi_{I,t}).$$

Here, the leverage limit $\bar{\lambda} \equiv \gamma^{-1} \exp(\xi_F - \xi_I)$ plausibly depends on diversification through the relative investment opportunities of financiers and insiders, captured by $\xi_F - \xi_I$. However, in a steady state in which the leverage constraint is conjectured to not bind for all other financiers (besides the one whose agency problem we are focusing on), simple calculations show $\xi_{F,\infty} - \xi_{I,\infty} = 0$. Intuitively, the long-run wealth distribution adjusts such that all agents earn the same idiosyncratic risk prices, which are the key determinants of ξ_F and ξ_I. Again, the leverage constraint is independent of diversification.

B.4 Optimal Choices for Log Utility Agents

In this section, I apply the convex duality approach of Cvitanić and Karatzas (1992) to solve agents' portfolio problems. This is a generalization of the martingale approach of Karatzas et al. (1987) and Cox and Huang (1989) to allow for portfolio constraints. I solve a slightly more general portfolio problem that nests the problems of insiders, financiers, and distressed investors.

Problem Setup

In general, all agents have a version of the following budget constraint:

$$dn_t = n_t[\mu^n_t dt + \sigma^n_t dZ_t + \hat{\sigma}^n_t d\hat{Z}_t], \quad n_0 > 0, \tag{26}$$

where Z and $\hat{Z}$ are two independent standard Brownian motions of dimensions D and M (Z is the vector of aggregate shocks), and

$$\begin{aligned} \mu^n_t &= r_t - \frac{c_t}{n_t} + \theta_t \pi_t + \lambda_t (a_t - r_t \mathbf{1}) \\ \sigma^n_t &= \theta_t + \lambda_t b_t \\ \hat{\sigma}^n_t &= \lambda_t \hat{b}_t. \end{aligned}$$

By appropriate definition of the variables $a_t \in \mathbb{R}^M$, $b_t \in \mathbb{R}^M \times \mathbb{R}^D$, and $\hat{b}_t \in \mathbb{R}^M \times \mathbb{R}^M$, equation (26) can replicate insiders', financiers', or distressed investors' net worth evolutions. For example, diversification Δ_A, Δ_B is accounted for by putting $\hat{b}_{F,t} = \mathrm{diag}(\hat{b}_{F,A,t}, \hat{b}_{F,B,t})$ with $\hat{b}_{F,z,t} = (1-\Delta_z)\hat{\sigma}_z$ and considering $\hat{Z}_t = (W^{A,\Delta}_{i,t}, W^{B,\Delta}_{i,t})'$, with $W^{z,\Delta}_{i,t} := (1-\Delta_z)^{-1}\Delta_z^{-1} \int_i^{i+\Delta_z} W^z_{j,t} dj$ a Brownian motion by Lemma 2.2.

In addition, we have the following portfolio constraints:

$$\lambda_t \in \Lambda \quad \text{and} \quad \theta_t \in \Theta \tag{27}$$

for $\Lambda := \{\lambda : \lambda \geq 0, \lambda \mathbf{1} \leq \bar{\lambda}\} \subset \mathbb{R}^M$ and either $\Theta := \mathbb{R}^D$ or $\Theta := \{0\}^D$. For insiders and distressed investors, $\bar{\lambda} = +\infty$. For insiders, in the case they cannot trade any aggregate risks, $\Theta = \{0\}^D$. When they can trade aggregate risks, $\Theta = \mathbb{R}^D$. This is always the case for financiers and distressed investors.

We are now in position to state agents' optimization problems, which are all sub-cases of the following. For $\mathcal{U}$ the logarithmic utility function defined in (1), agents solve

$$\mathcal{U}_t^* := \sup_{n,c,\lambda,\theta} \mathcal{U}_t \tag{28}$$

subject to (26), (27), and $n_t \geq 0$.

Heuristic Derivation of Dual Static Optimization Problem

The heuristic derivation of optimal controls is as follows. The necessary technical arguments are presented in Cvitanić and Karatzas (1992). For any convex set $\mathcal{A}$, define the penalty function

$$\varphi_{\mathcal{A}}(x) := \begin{cases} 0, & \text{if} \quad x \in \mathcal{A}; \\ -\infty, & \text{if} \quad x \notin \mathcal{A}. \end{cases}$$

Augment the wealth dynamics by φ_Λ and φ_Θ to account for the portfolio constraints (27):

$$dn_t = n_t[\mu_t^n dt + \varphi_\Lambda(\lambda_t)dt + \varphi_\Theta(\theta_t)dt + \sigma_t^n dZ_t + \hat{\sigma}_t^n d\hat{Z}_t].$$

Introduce an Itô process (which will represent the state-price density or Lagrange multiplier process):

$$d\xi_t = -\xi_t \left[\alpha_t dt + \beta_t \cdot dZ_t + \hat{\beta}_t \cdot d\hat{Z}_t\right]. \tag{29}$$

Itô's formula implies

$$\begin{aligned} \xi_\tau n_\tau = \xi_0 n_0 &+ \int_0^\tau \xi_t n_t \Big(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t \Big) dt \\ &+ \int_0^\tau \xi_t n_t \Big(-\beta_t' + \sigma_t^n \Big) dZ_t + \int_0^\tau \xi_t n_t \Big(-\hat{\beta}_t' + \hat{\sigma}_t^n \Big) d\hat{Z}_t. \end{aligned} \tag{30}$$

Now, we want to take expectations to eliminate the stochastic integrals, and then to take $\tau \to +\infty$. Doing this requires a series of technical arguments.

First, τ may be a stopping time rather than a deterministic time. In particular, the equilibrium of the model will imply, in principal, that $-r_t$ and π_t can be arbitrarily large, so I localize the integral with $\tau \equiv \tau_L := T \wedge \tau_L^{-r} \wedge \tau_L^{-\alpha} \wedge \tau_L^{\pi} \wedge \tau_L^{\beta} \wedge \tau_L^{\hat{\beta}}$, where $T > 0$ is deterministic and for any process x we have defined $\tau_L^x := \inf\{t \geq 0 : x_t \geq L\}$ for some $L > 0$. However, equilibrium will have the property that $\lim_{L\to\infty} \tau_L = T$ almost-surely, because the probability of large $-r_t$ or π_t vanishes (this can be verified ex-post using the equilibrium state dynamics from Proposition C.3). Consequently, we may take expectations, followed by the limit $L \to +\infty$, to obtain

$$\mathbb{E}[\xi_T n_T] = \xi_0 n_0 + \mathbb{E} \int_0^T \xi_t n_t \Big(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t \Big) dt,$$

where τ is replaced with T inside the expectations by the dominated convergence theorem, which holds as long as $\lambda \in \Lambda$ and $\theta \in \Theta$. Indeed, the coefficients of ξ and n are uniformly bounded up to time T.

Because maximization will imply a transversality condition on discounted wealth, assume $\lim_{T\to\infty} \xi_T n_T = 0$ almost-surely. The transversality condition will have to be verified in equilibrium. If it holds, then we may apply appropriate convergence theorems to take $T \to +\infty$. Indeed, we may split $\xi_t n_t(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t)$ into positive and negative parts and apply the monotone convergence theorem separately to the integrals of these parts. Furthermore, by ignoring the negative part, we have $\mathbb{E}[\xi_T n_T] \leq \xi_0 n_0 + \mathbb{E} \int_0^\infty \xi_t n_t(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t)^+ dt$. This upper bound implies we may apply the dominated convergence theorem to take $\lim_{T\to\infty} \mathbb{E}[\xi_T n_T] = 0$. The result of taking these limits is

$$0 = \xi_0 n_0 + \mathbb{E} \int_0^\infty \xi_t n_t \Big(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \cdot \beta_t - \hat{\sigma}_t^n \cdot \hat{\beta}_t \Big) dt. \tag{31}$$

The "static" budget constraint (31) is an implication of the dynamic wealth constraint (30), which means that the result of the unconstrained problem

$$\sup_{n\geq 0,c,\lambda,\theta} \mathbb{E}\Big[\int_0^\infty \Big(\rho e^{-\rho t}\log c_t + \xi_t n_t\Big(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t\Big)\Big)dt + \xi_0 n_0\Big] \tag{32}$$

is technically an upper bound on the maximized constrained objective (28). The point of Cvitanić and Karatzas (1992), Theorem 10.1, is to show that by minimizing over the process ξ in (29), one can obtain the value of the maximized constrained objective, i.e.,

$$\mathcal{U}_0^* = \inf_{\xi} \sup_{n\geq 0,c,\lambda,\theta} \mathbb{E}\Big[\int_0^\infty \Big(e^{-\rho t}\log c_t + \xi_t n_t\Big(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t\Big)\Big)dt + \xi_0 n_0\Big]. \tag{33}$$

Furthermore, the order of minimization and maximization may be exchanged. With this equivalence, optimal policies can be found from the unconstrained problem (32) for *some* process ξ that is suitably minimal.

Solving the Static Problem

First, we solve the maximization problem. The first-order condition with respect to c is typical:

$$\rho e^{-\rho t}\frac{1}{c_t} = \xi_t. \tag{34}$$

To solve for optimal portfolios, introduce the "slackness" processes

$$\nu := b\beta + \hat{b}\hat{\beta} + r\mathbf{1} - a \tag{35}$$
$$\omega := \beta - \pi. \tag{36}$$

Maximizing over (λ,θ) are thus equivalent to maximizing $\varphi_\Lambda(\lambda) - \lambda\nu$ and $\varphi_\Theta(\theta) - \theta\omega$. With that in mind, for any convex set $\mathcal{A}$ define the convex support function $\tilde{\varphi}_\mathcal{A}(x) := \sup_y \varphi_\mathcal{A}(y) - yx = \sup_{y\in\mathcal{A}}(-yx)$. Defining $\bar{\nu} := \min(\nu)$, these conjugate functions are given by (regardless of whether $\Theta = \mathbb{R}^D$ or $\Theta = \{0\}^D$)

$$\tilde{\varphi}_\Lambda(\nu) = \bar{\lambda}\max(0, -\bar{\nu}) \tag{37}$$
$$\tilde{\varphi}_\Theta(\omega) = 0. \tag{38}$$

Note that, when $\Theta = \mathbb{R}^D$, it must be the case that $\omega_t \equiv 0$. Finally, substituting (34), (37), and (38) back into the objective (32), we have

$$\sup_{n\geq 0} \mathbb{E}\Big[\int_0^\infty \Big(\rho e^{-\rho t}[-1 + \log\rho - \rho t - \log\xi_t] + \xi_t n_t[-\alpha_t + r_t + \bar{\lambda}\max(0, -\bar{\nu}_t)]\Big)dt + \xi_0 n_0\Big].$$

Assuming $n_t > 0$ for all t (which can be verified ex-post by the optimal wealth dynamics), maximizing over n implies that

$$-\alpha_t + r_t + \max(0, -\bar{\lambda}\bar{\nu}_t) = 0. \tag{39}$$

Next, minimizing over ξ in (33) amounts to minimizing over (ν,ω) and the initial value ξ_0. This is because the coefficients of ξ depend on market prices and the process (ν,ω), as seen in the necessary conditions (35), (36), and (39). To emphasize this dependence, write $\xi^{\nu,\omega}$, $\alpha^{\nu,\omega}$, $\beta^{\nu,\omega}$, and $\hat{\beta}^{\nu,\omega}$ for the Lagrange multiplier process and its coefficient processes. In particular, note that

$$\xi_t^{\nu,\omega} = \xi_0 \exp\Big\{-\int_0^t \Big(\alpha_s^{\nu,\omega} + \frac{1}{2}\|\beta_s^{\nu,\omega}\|^2 + \frac{1}{2}\|\hat{\beta}_s^{\nu,\omega}\|^2\Big)ds - \int_0^t \beta_s^{\nu,\omega}\cdot dZ_s - \int_0^t \hat{\beta}_s^{\nu,\omega}\cdot d\hat{Z}_s\Big\} \tag{40}$$
$$\alpha_t^{\nu,\omega} = r_t + \bar{\lambda}\max(0, -\bar{\nu}_t) \tag{41}$$
$$\beta_t^{\nu,\omega} = \omega_t + \pi_t \tag{42}$$
$$\hat{\beta}_t^{\nu,\omega} = \hat{b}_t^{-1}[a_t - r_t\mathbf{1} - b_t(\omega_t + \pi_t) + \nu_t] \tag{43}$$

We are led to solve the dual problem

$$\inf_{\nu,\omega,\xi_0} -\mathbb{E}\Big[\int_0^\infty \rho e^{-\rho t}\log \xi_t^{\nu,\omega} dt - \xi_0 n_0\Big], \tag{44}$$

subject to (40), (41), (42), (43), and additionally $\omega_t = 0$ if we set $\Theta = \mathbb{R}^D$.

Substituting $\xi^{\nu,\omega}$ into the objective (44), we immediately solve for the initial condition and find that

$$\xi_0 = \frac{1}{n_0}. \tag{45}$$

Then, assuming we can perform appropriate localizations on the stochastic integrals in (40) as before, the processes (ν,ω) are determined from solving

$$\inf_{\nu,\omega}\mathbb{E}\Big[\int_0^\infty \rho e^{-\rho t}\int_0^t \Big(r_s + \bar{\lambda}\max(0,-\bar{\nu}_s) + \frac{1}{2}\|\omega_s+\pi_s\|^2 + \frac{1}{2}\|\hat{b}_s^{-1}(a_s - r_s\mathbf{1} - b_s(\omega_s+\pi_s)+\nu_s)\|^2\Big)dsdt\Big].$$

Crucially, notice that the minimization can be taken pointwise, i.e.,

$$\omega_t = \arg\min_{x\in\mathbb{R}^D}\Big\{\frac{1}{2}\|x+\pi_t\|^2 + \frac{1}{2}\|\hat{b}_t^{-1}(a_t - r_t\mathbf{1} - b_t(x+\pi_t)+\nu_t)\|^2\Big\}$$

and

$$\nu_t = \arg\min_{x\in\mathbb{R}^M}\Big\{\bar{\lambda}\max(0,-\min(x)) + \frac{1}{2}\|\hat{b}_t^{-1}(a_t - r_t\mathbf{1} - b_t(\omega_t+\pi_t)+x)\|^2\Big\}.$$

These are convex problems and have unique solutions. For reference, these are the same as equation (11.4) in Cvitanić and Karatzas (1992).

Recall that $\omega_t = 0$ if $\Theta = \mathbb{R}^D$. If $\Theta = \{0\}^D$ instead, then by inspection we see that $\omega_t = -\pi_t$ is the optimal choice.

Now we solve for ν. In all of the applications in the paper, $\hat{b}$ is a diagonal matrix with dimension $M = 2$. To solve this problem, I specialize to this case, which simplifies the calculations. Now

$$\|\hat{b}^{-1}(a - r\mathbf{1} - b(\omega+\pi)+\nu)\|^2 = \sum_{i=1}^2\Big(\frac{[a]_i - r - [b(\omega+\pi)]_i + [\nu]_i}{[\hat{b}]_{ii}}\Big)^2,$$

where $[x]_i$ and $[y]_{ij}$ represent the ith element of the vector x and (i,j)th element of the matrix y. Define $\hat{\pi}_i := [\hat{b}]_{ii}^{-2}([a]_i - r - [b(\omega+\pi)]_i)$. Minimizing with respect to ν requires a case-by-case analysis, similar to example 14.9 in Cvitanić and Karatzas (1992):

- $\hat{\pi}_1 \le 0, \hat{\pi}_2 \le 0$.
  Optimal choice: $[\nu]_1 = -[\hat{b}]_{11}^2\hat{\pi}_1$ and $[\nu]_2 = -[\hat{b}]_{22}^2\hat{\pi}_2$.
  Rationale: If either $\hat{\pi}_i \le 0$, the optimal choice for $[\nu]_i$ can be made independent of $[\nu]_{-i}$ and that choice is $[\nu]_i = -[\hat{b}]_{ii}^2\hat{\pi}_i$.
- $\hat{\pi}_1 > 0, \hat{\pi}_2 \le 0$.
  Optimal choice: $[\nu]_1 = -[\hat{b}]_{11}^2(\hat{\pi}_1 - \bar{\lambda})^+$ and $[\nu]_2 = -[\hat{b}]_{22}^2\hat{\pi}_2$.
  Rationale: $[\nu]_2$ can be chosen according to the previous case.
  If $\hat{\pi}_1 > \bar{\lambda}$, then the choice of $[\nu]_1 = -[\hat{b}]_{11}^2(\hat{\pi}_1 - \bar{\lambda}) = \bar{\nu} < 0$ minimizes $-\bar{\lambda}\bar{\nu} + \frac{1}{2}[\hat{b}]_{11}^{-2}([\hat{b}]_{11}^2\hat{\pi}_1 + \bar{\nu})^2$.
  If $\hat{\pi}_1 \le \bar{\lambda}$, then it must be that $\bar{\nu} \ge 0$ in which case $[\nu]_1 = 0$ minimizes $\frac{1}{2}[\hat{b}]_{11}^{-2}([\hat{b}]_{11}^2\hat{\pi}_1 + [\nu]_1)^2$.
- $\hat{\pi}_1 \le 0, \hat{\pi}_2 > 0$.
  Optimal choice: $[\nu]_1 = -[\hat{b}]_{11}^2\hat{\pi}_1$ and $[\nu]_2 = -[\hat{b}]_{22}^2(\hat{\pi}_2 - \bar{\lambda})^+$.
  Rationale: This case is symmetrical to the previous one.

- $\hat{\pi}_1 > 0, \hat{\pi}_2 > 0$ and $\hat{\pi}_1 + \hat{\pi}_2 \leq \bar{\lambda}$.
  Optimal choice: $[\nu]_1 = 0$ and $[\nu]_2 = 0$.
  Rationale: Choosing either (or both) $[\nu]_1, [\nu]_2 < 0$ is not feasible because first-order optimality cannot be satisfied. Moreover, choosing $[\nu]_1, [\nu]_2 > 0$ is not optimal, leaving the zero solution.
- $\hat{\pi}_1 > 0, \hat{\pi}_2 > 0$ and $\hat{\pi}_1 + \hat{\pi}_2 > \bar{\lambda}$.
  Here, it must be the case that $\nu \leq 0$ with at least one of $[\nu]_1, [\nu]_2$ strictly negative. Consider the three sub-cases $[\nu]_1 < [\nu]_2 \leq 0$, $[\nu]_2 < [\nu]_1 \leq 0$, and $[\nu]_1 = [\nu]_2 < 0$. In the first case, the optimal choices are $[\nu]_1 = [\hat{b}]_{11}^2[\bar{\lambda} - \hat{\pi}_1]$ and $[\nu]_2 = -[\hat{b}]_{22}^2\hat{\pi}_2$, and these two must be ordered as anticipated. The second case is symmetrical. The third case with $[\nu]_1 = [\nu]_2 = \bar{\nu}$ has the optimality condition $\bar{\nu} = ([\hat{b}]_{11}^{-2} + [\hat{b}]_{22}^{-2})^{-1}[\bar{\lambda} - \hat{\pi}_1 - \hat{\pi}_2]$. Thus, we have the three corresponding sub-cases.

  * $\hat{\pi}_1 - [\hat{b}]_{11}^{-2}[\hat{b}]_{22}^2\hat{\pi}_2 \geq \bar{\lambda}$.
    Optimal choice: $[\nu]_1 = [\hat{b}]_{11}^2[\bar{\lambda} - \hat{\pi}_1]$ and $[\nu]_2 = -[\hat{b}]_{22}^2\hat{\pi}_2$.
  * $\hat{\pi}_2 - [\hat{b}]_{22}^{-2}[\hat{b}]_{11}^2\hat{\pi}_1 \geq \bar{\lambda}$.
    Optimal choice: $[\nu]_1 = -[\hat{b}]_{11}^2\hat{\pi}_1$ and $[\nu]_2 = [\hat{b}]_{22}^2[\bar{\lambda} - \hat{\pi}_2]$.
  * $\bar{\lambda} > \max\{\hat{\pi}_1 - [\hat{b}]_{11}^{-2}[\hat{b}]_{22}^2\hat{\pi}_2, \hat{\pi}_2 - [\hat{b}]_{22}^{-2}[\hat{b}]_{11}^2\hat{\pi}_1\}$.
    Optimal choice: $[\nu]_1 = [\nu]_2 = ([\hat{b}]_{11}^{-2} + [\hat{b}]_{22}^{-2})^{-1}[\bar{\lambda} - \hat{\pi}_1 - \hat{\pi}_2]$.

Consumption and Portfolios

Now, we use the solution of the dual problem to determine optimal policies. First, substitute the optimality conditions (35), (36), (37), (38), and (39) into the time-t version of the static budget constraint (31), which shows that optimal wealth is given by

$$\xi_t^{\nu,\omega} n_t = \mathbb{E}_t\Big[\int_0^\infty \xi_{t+s}^{\nu,\omega} c_{t+s} ds\Big].$$

Using (34), we obtain the familiar log utility consumption rule $c_t = \rho n_t$. Second, substitute these optimality conditions, and (45), into the dynamic budget constraint (30) to obtain

$$0 = \int_0^T \xi_t n_t(-(\beta_t^{\nu,\omega})' + \sigma_t^n) dZ_t + \int_0^T \xi_t n_t(-(\hat{\beta}_t^{\nu,\omega})' + \hat{\sigma}_t^n) d\hat{Z}_t, \quad \text{i.e.,} \quad (\sigma_t^n)' = \beta_t^{\nu,\omega} \quad \text{and} \quad (\hat{\sigma}_t^n)' = \hat{\beta}_t^{\nu,\omega}.$$

Thus, using our explicit solution for ν in the $M = 2$ case with $\hat{b}$ a diagonal matrix, the optimal λ is determined as

$$\lambda_t = \hat{b}_t^{-1}\hat{\beta}_t^{\nu,\omega} = \hat{b}_t^{-2}(a_t - r_t\mathbf{1} - b_t(\omega_t + \pi_t) - \zeta_t)^+,$$

where $\zeta_t := -\max(0, -\bar{\nu}_t)$. This is the generalization of equations (20) and (18), which are obtained using $\omega_t = 0$ (since financiers and distressed investors can access aggregate Arrow markets without constraints) and substituting appropriate $a, b, \hat{b}$ from the model. One can verify that $\zeta_t > 0$ only when $\hat{b}_t^{-2}(a_t - r_t\mathbf{1} - b_t(\omega_t + \pi_t))^+ \cdot \mathbf{1} \geq \bar{\lambda}$. Given our formula for λ_t and this observation, this verifies the complementary slackness formula (19).

B.5 Optimal Choices for Recursive Utility Agents

In some extensions, I will want to consider more general preferences than log, which requires a dynamic programming method, unlike Appendix B.4. Relatedly, to analyze mobility decisions under Assumption 2, it is important to have agents' dynamic programming equations.

Suppose agents' have recursive Duffie and Epstein (1992) utility recursions, given by

$$\mathcal{U}_t := \mathbb{E}_t\Big[\int_t^\infty \varphi(c_s, \mathcal{U}_s) ds\Big],$$
$$\text{where} \quad \varphi(c, \mathcal{U}) := \frac{\rho(1-\gamma)\mathcal{U}}{1-\varsigma}\Big(c^{1-\varsigma}[(1-\gamma)\mathcal{U}]^{-\frac{1-\varsigma}{1-\gamma}} - 1\Big). \tag{46}$$

In (46), $\rho > 0$ represents the subjective discount rate, $\gamma > 0$ represents the coefficient of relative risk aversion (RRA), and $\varsigma^{-1} > 0$ represents the elasticity of intertemporal substitution (EIS). Setting $\varsigma = \gamma$, these preferences reduce to von Neumann-Morgenstern preferences. Setting $\varsigma = 1$, the utility aggregator function ς becomes logarithmic over the consumption bundle.[39] Then, as in Appendix B.4 all agents' portfolio problems can be written as

$$\max_{n,c,\theta,\lambda} \mathcal{U}_t \tag{47}$$

subject to (26), $n_t \geq 0$, $\lambda_t \in \Lambda$, and $\theta_t \in \Theta$ for closed, convex sets $\Lambda \subset \mathbb{R}^M$ and $\Theta \subset \mathbb{R}^D$. To simplify exposition, I assume $\Lambda = \{\lambda : \lambda \geq 0, \lambda\mathbf{1} \leq \bar{\lambda}\}$ as in (27) and $\Theta = (\underline{\theta}_1, \bar{\theta}_1) \times \cdots \times (\underline{\theta}_D, \bar{\theta}_D)$ for $\underline{\theta} := (\underline{\theta}_1, \ldots, \underline{\theta}_D) \in \mathbb{R}^D_- \cup \{-\infty\}^D$ and $\bar{\theta} := (\bar{\theta}_1, \ldots, \bar{\theta}_D) \in \mathbb{R}^D_+ \cup \{-\infty\}^D$. This assumption on Θ generalizes (27). All agents except financiers have $\bar{\lambda} = +\infty$. Whenever agents can freely trade aggregate risk in Arrow markets (e.g., both financiers and distressed investors can always do this), we have $\underline{\theta} = \{-\infty\}^D$ and $\bar{\theta} = \{+\infty\}^D$. When agents cannot trade at all in these markets (e.g., in the model of Section 3, insiders cannot trade), we have $\underline{\theta} = \bar{\theta} = \{0\}^D$.

To solve (47), we first use its scaling properties to simplify the problem. Given the homotheticity of preferences combined with the linearity of wealth evolution, value functions take the form

$$\mathcal{U}_t = \frac{(n_t \xi_t)^{1-\gamma}}{1-\gamma},$$

where

$$d\xi_t = \xi_t \Big[\mu^\xi_t dt + \sigma^\xi_t dZ_t\Big]. \tag{48}$$

The process ξ_t represents the investment opportunity set of the agent and responds only to the aggregate shock Z, due to the free mobility condition, Assumption 2.[40]

Then, the HJB equation of such an agent is given by

$$0 = \max_{c,\lambda\in\Lambda,\theta\in\Theta} \Big\{\varphi(c,\mathcal{U}) + n\mu^n \partial_n \mathcal{U} + \frac{1}{2}n^2[\|\sigma^n\|^2 + \|\hat{\sigma}^n\|^2]\partial_{nn}\mathcal{U} + \xi\mu^\xi\partial_\xi\mathcal{U} + \frac{1}{2}\xi^2\|\sigma^\xi\|^2\partial_{\xi\xi}\mathcal{U} + n\xi\sigma^n(\sigma^\xi)'\partial_{n\xi}\mathcal{U}\Big\}.$$

Substituting the form of $\mathcal{U}$ and its derivatives, then dividing the entire HJB equation by the positive quantity $(n\xi)^{1-\gamma}$, we obtain

$$0 = \max_{c,\lambda\in\Lambda,\theta\in\Theta} \Big\{\rho\frac{(\frac{c}{n\xi})^{1-\varsigma} - 1}{1-\varsigma} + \mu^n - \frac{\gamma}{2}[\|\sigma^n\|^2 + \|\hat{\sigma}^n\|^2] + \mu^\xi - \frac{\gamma}{2}\|\sigma^\xi\|^2 + (1-\gamma)\sigma^n(\sigma^\xi)'\Big\},$$

First-order optimality for this agent implies for consumption:

$$c_t = \rho^{1/\varsigma}\xi_t^{1-1/\varsigma} n_t \tag{49}$$

Optimal portfolios must satisfy the following complementary slackness conditions:

$$0 = \min\{\lambda', -a + (r+\zeta)\mathbf{1} + \gamma b(\sigma^n)' + (\gamma-1)b(\sigma^\xi)' + \gamma\hat{b}(\hat{\sigma}^n)'\} \tag{50}$$

$$0 = \min\{\zeta, \bar{\lambda} - \lambda\mathbf{1}\} \tag{51}$$

$$0 = \max\{\theta' - \bar{\theta}', \min\{\theta' - \underline{\theta}', -\pi + \gamma(\sigma^n)' + (\gamma-1)(\sigma^\xi)'\}\}. \tag{52}$$

[39] By taking the limit $\varsigma \to 1$ with L'Hôpital's rule, the aggregator becomes

$$\varphi(c,\mathcal{U}) = \rho(1-\gamma)\mathcal{U}\Big[\log(c) - \frac{1}{1-\gamma}\log[(1-\gamma)\mathcal{U}]\Big].$$

[40] Verifying this equilibrium property is straightforward. Indeed, if ξ_t were affected by idiosyncratic shocks W, then different locations would have different levels of ξ_t. Free mobility implies agents would immediately migrate to locations with higher levels of ξ_t and attain a higher value function, which is a contradiction.

Plugging these choices back into the HJB equation, we obtain the following:

$$0 = \rho \frac{\varsigma(\xi/\rho)^{\frac{\varsigma-1}{\varsigma}} - 1}{1-\varsigma} + r + \bar{\lambda}\zeta + \theta\Big[\pi - \gamma(\sigma^n)' - (\gamma-1)(\sigma^\xi)'\Big] + \frac{\gamma}{2}\Big[\|\sigma^n\|^2 + \|\hat{\sigma}^n\|^2\Big] + \mu^\xi - \frac{\gamma}{2}\|\sigma^\xi\|^2, \tag{53}$$

where θ and ζ are determined using conditions (50)-(52). Note that $\rho\frac{\varsigma(\xi/\rho)^{1-1/\varsigma}-\rho}{1-\varsigma} \to \rho(\log(\rho/\xi) - 1)$ as $\varsigma \to 1$. Because ξ will be a function of aggregate state variables in a Markovian equilibrium, μ^ξ and σ^ξ may be determined in terms of the derivatives of ξ by Itô's formula. Thus, (53) is a differential equation for ξ. In principle, one could develop an infinite-horizon extension of the "verification"-type arguments of Schroder and Skiadas (2003), which nests the choice problem above aside from the finite horizon. This would show that solving equation (53) is sufficient for optimality of the choices outlined above.

Appendix B.4 proves the convex duality approach yields exactly these optimality conditions for log utility. There is no need to solve (53) in this case.

C Equilibrium Proofs and Derivations

C.1 Equilibrium for Section 2

Proof of Proposition 2.3. This is a special case of Proposition C.3 with $\bar{\lambda} = +\infty$, $\chi = 0$, $\delta = 0$, $\rho_F = \rho$, and $\sigma_A = \sigma_B \equiv 0$. Substituting $\zeta = \chi = 0$ into equation (66), and then substituting the result into equation (69), we obtain equation (14). Similarly, $\chi = 0$ and $\rho_F = \rho$ in equation (21), we obtain an equation for ι. By time-differentiating the goods market clearing condition, we obtain the following equation for r:

$$r = \rho + \iota - (1-\eta)[\alpha \hat{\pi}_A^2 + (1-\alpha)\hat{\pi}_B^2] - \eta[\hat{\pi}_{F\to A}^2 + \hat{\pi}_{F\to B}^2]. \tag{54}$$

The state dynamics of (α, η) are obtained by substituting $\delta = 0$ and $\rho_F = \rho$ and $\pi_A = \pi_B = \pi = 0$.

Existence/uniqueness follows from uniqueness of optimal choices from Appendix B.4, the explicit solution (14) for κ_t, and the explicit solutions for all other equilibrium objects, conditional on (α, η, κ). $\square$

Next, consider the long-run equilibrium. If a steady state exists, it must satisfy $\mu^\alpha = \mu^\eta = 0$. Supposing κ_∞ is the steady-state capital share, the solution to this system is

$$\alpha_\infty := \frac{\kappa_\infty(1-\phi_A)\hat{\sigma}_A}{\kappa_\infty(1-\phi_A)\hat{\sigma}_A + (1-\kappa_\infty)(1-\phi_B)\hat{\sigma}_B} \tag{55}$$

$$\eta_\infty := \frac{\sqrt{(\kappa_\infty\phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-\kappa_\infty)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2}}{\sqrt{(\kappa_\infty\phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-\kappa_\infty)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2} + \kappa_\infty(1-\phi_A)\hat{\sigma}_A + (1-\kappa_\infty)(1-\phi_B)\hat{\sigma}_B}. \tag{56}$$

We have the following.

Proposition C.1 (Steady State). *Let $\hat{\sigma}_A > 0, \hat{\sigma}_B > 0, \phi_A \in (0,1), \phi_B \in (0,1), \Delta_A \in (0,1), \Delta_B \in (0,1)$ and suppose $|G_A - G_B|$ and $|(1-\phi_A)\hat{\sigma}_A - (1-\phi_B)\hat{\sigma}_B|$ are sufficiently small. Given initial wealth shares $\alpha_0, \eta_0 > 0$, there exists a steady state given by $(\alpha_\infty, \eta_\infty)$ in equations (55)-(56), where κ_∞ is given by the time-limit of equation (14).*

Proof of Proposition C.1. Solving $\mu^\alpha = \mu^\eta = 0$, conditional on $\kappa = y$, defines functions $\alpha^*(\kappa)$ and $\eta^*(\kappa)$:

$$\alpha^*(y) := \frac{y(1-\phi_A)\hat{\sigma}_A}{y(1-\phi_A)\hat{\sigma}_A + (1-y)(1-\phi_B)\hat{\sigma}_B}$$

$$\eta^*(y) := \frac{\sqrt{(y\phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-y)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2}}{\sqrt{(y\phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-y)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2} + y(1-\phi_A)\hat{\sigma}_A + (1-y)(1-\phi_B)\hat{\sigma}_B}.$$

Similarly, equation (14), which holds for any values of (α, η), defines a function $\kappa^*(\alpha, \eta)$. Then, for $y \in [0,1]$ define $F(y) := \kappa^*(\alpha^*(y), \eta^*(y)) - y$. To prove a steady state exists, it suffices to prove that $F(y) = 0$ has a root in $[0,1]$; call it κ_∞. In that case, an interior steady state is given by $\alpha_\infty = \alpha^*(\kappa_\infty)$ and $\eta_\infty = \eta^*(\kappa_\infty)$, given in (55)-(56).

The function $F(y)$ is given by

$$F(y) := -y + \min\{1, \max\{0, \tilde{\kappa}^*(y)\}\} \tag{57}$$

$$\text{where} \quad \tilde{\kappa}^*(y) := \frac{G_A - G_B + M_B(y)}{M_A(y) + M_B(y)}, \tag{58}$$

and where $M_A(y) := [\frac{(1-\phi_A)^2}{\alpha^*(y)(1-\eta^*(y))} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta^*(y)}]\hat{\sigma}_A^2$ and $M_B(y) := [\frac{(1-\phi_B)^2}{(1-\alpha^*(y))(1-\eta^*(y))} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta^*(y)}]\hat{\sigma}_B^2$. By inspection, $F(0) = F(1) = 0$. We aim to show $F'(0+)$ and $F(1-)$ have the same sign, which proves the claim by continuity of F. If $|G_A = G_B|$ is small enough, then $\tilde{\kappa}^*(y) \in (0,1)$ for all y. Given this, one may then differentiate F to compute

(keeping only non-vanishing terms)

$$
\begin{aligned}
F'(0+) &= -1 - \lim_{y\to 0}(G_A - G_B + M_B(y))\frac{M_A'(y)}{M_A(y)^2} \\
&= -1 + (G_A - G_B)\frac{1-\eta^*(0)}{(1-\phi_A)(1-\phi_B)\hat{\sigma}_A\hat{\sigma}_B} + \frac{(1-\phi_B)\hat{\sigma}_B}{(1-\phi_A)\hat{\sigma}_A} + \frac{1-\eta^*(0)}{\eta^*(0)}\frac{\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}{(1-\phi_A)(1-\phi_B)\hat{\sigma}_A\hat{\sigma}_B} \\
F'(1-) &= -1 - \lim_{y\to 1}(G_A - G_B - M_A(y))\frac{M_B'(y)}{M_B(y)^2} \\
&= -1 - (G_A - G_B)\frac{1-\eta^*(1)}{(1-\phi_A)(1-\phi_B)\hat{\sigma}_A\hat{\sigma}_B} + \frac{(1-\phi_A)\hat{\sigma}_A}{(1-\phi_B)\hat{\sigma}_B} + \frac{1-\eta^*(1)}{\eta^*(1)}\frac{\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2}{(1-\phi_A)(1-\phi_B)\hat{\sigma}_A\hat{\sigma}_B}.
\end{aligned}
$$

Using the expressions $\eta^*(0) = \frac{\phi_B(1-\Delta_B)}{1-\phi_B+\phi_B(1-\Delta_B)}$ and $\eta^*(1) = \frac{\phi_A(1-\Delta_A)}{1-\phi_A+\phi_A(1-\Delta_A)}$, and simplifying, we obtain

$$
\begin{aligned}
F'(0+) &= (G_A - G_B)\frac{1-\eta^*(0)}{(1-\phi_A)(1-\phi_B)\hat{\sigma}_A\hat{\sigma}_B} + \frac{(1-\phi_B)\hat{\sigma}_B - (1-\phi_A)\hat{\sigma}_A + \phi_B(1-\Delta_B)\hat{\sigma}_B}{(1-\phi_A)\hat{\sigma}_A} \\
F'(1-) &= (G_A - G_B)\frac{1-\eta^*(1)}{(1-\phi_A)(1-\phi_B)\hat{\sigma}_A\hat{\sigma}_B} + \frac{(1-\phi_A)\hat{\sigma}_A - (1-\phi_B)\hat{\sigma}_B + \phi_A(1-\Delta_A)\hat{\sigma}_A}{(1-\phi_B)\hat{\sigma}_B}.
\end{aligned}
$$

Thus, when $|(1-\phi_A)\hat{\sigma}_A - (1-\phi_B)\hat{\sigma}_B|$ and $|G_A - G_B|$ are small enough, $F'(0+) > 0$ and $F(1-) > 0$. □

C.2 Impulse Response Characterization

Define the IRF of a stationary variable Y by

$$
\mathcal{I}[Y](t, x; \Delta) := \mathbb{E}[Y_{\tau+t} - Y_{\tau-} \mid X_{\tau-} = x], \quad t \geq 0, \tag{59}
$$

where X_t is the vector of state variables, i.e., $X_t = (\alpha_t, \eta_t)$ in Section 2, and Δ is the variable receiving an unanticipated shock at time τ, i.e., $\Delta_\tau \neq \Delta_{\tau-}$. Equation (59) can be decomposed into the sum of an "impact response" $\mathbb{E}[Y_\tau - Y_{\tau-} \mid X_{\tau-} = x]$ and a "transition path" $\mathbb{E}[Y_{\tau+t} - Y_\tau \mid X_\tau = x']$.

In the baseline model of Section 2, we first considered one-time unanticipated shocks to (Δ_A, Δ_B). An important simplifying property of this model is that these shocks do not generate any impact response to the state variables in the model, as stated in Lemma 2.5. Thus, $\mathcal{I}[X](t, x; \Delta) = \mathbb{E}[X_{\tau+t} - X_\tau \mid X_\tau = x]$ for this type of shock. The top panels of figure 6 in the main text does this IRF analysis for a one-time shock to Δ_A. The bottom panels repeat the analysis for a gradual increase in Δ_A, which may be interpreted in three equivalent ways – fully unanticipated, fully anticipated, or partially anticipated – in the sense of Lemma 2.4.

Proof of Lemma 2.4. Part (i) follows directly from Lemma 2.5. Part (ii) is a special case of part (iii) for $t \neq \tau$. For $t = \tau$, we make use of Proposition 2.5. Part (iii) follows from the fact that optimal choices of all log utility agents in the model are independent of the Itô processes driving expected returns and volatilities, see Appendix B.4. Hence, Proposition C.3 still holds even in this more general case. The equilibrium of Proposition 2.3 follows as a special case of Proposition C.3, as before. □

Proof of Lemma 2.5. We want to show that the impact responses $\mathbb{E}[\eta_\tau - \eta_{\tau-} \mid X_{\tau-} = x] = \mathbb{E}[\alpha_\tau - \alpha_{\tau-} \mid X_{\tau-} = x] = 0$ in response to a shock to Δ_A at time τ. A similar analysis holds for state variable α and for a shock to Δ_B. Note that aggregated net worths can be written as

$$
\begin{aligned}
N_{F,t} &= [\kappa_t\phi_A + (1-\kappa_t)\phi_B]K_t + M_{F,t} + p_t\tilde{\theta}_{F,t}N_{F,t} \\
N_{A,t} &= \kappa_t(1-\phi_A)K_t + M_{A,t} + p_t\tilde{\theta}_{A,t}N_{A,t} \\
N_{B,t} &= (1-\kappa_t)(1-\phi_B)K_t + M_{B,t} + p_t\tilde{\theta}_{B,t}N_{B,t},
\end{aligned}
$$

where $M_{z,t}$ are the riskless assets, $\tilde{\theta}_{z,t}N_{z,t}$ are the hedging securities (Arrow securities on the aggregate shock) held, and p_t the price of these securities. Bond market clearing is that $M_{F,t} + M_{A,t} + M_{B,t} = 0$. Notice that equity market clearing is already imposed. Since the Arrow securities pay an excess return, their price is zero, i.e., $p_t = 0$ for all t. We can thus rewrite the net worth equations immediately prior to the shock as

$$\begin{aligned}
N_{F,\tau-} &= [\kappa_{\tau-}\phi_A + (1-\kappa_{\tau-})\phi_B]K_{\tau-} - M_{A,\tau-} - M_{B,\tau-} \\
N_{A,\tau-} &= \kappa_{\tau-}(1-\phi_A)K_{\tau-} + M_{A,\tau-} \\
N_{B,\tau-} &= (1-\kappa_{\tau-})(1-\phi_B)K_{\tau-} + M_{B,\tau-}.
\end{aligned}$$

Upon the shock, the variables $K, \kappa, 1-\kappa, M_A, M_B$ cannot jump. Indeed, a jump in K cannot be consistent with goods market clearing, whereas M_A, M_B represent trading positions (asset holdings) and are assumed fixed at the time of the shock. Furthermore, the Arrow securities are settled every dt time periods, so they have no continuation value that could possibly jump. Then, after the shock,

$$\begin{aligned}
N_{F,\tau} &= [\kappa_{\tau-}\phi_A + (1-\kappa_{\tau-})\phi_B]K_{\tau-} - M_{A,\tau-} - M_{B,\tau-} \\
N_{A,\tau} &= \kappa_{\tau-}(1-\phi_A)K_{\tau-} + M_{A,\tau-} \\
N_{B,\tau} &= (1-\kappa_{\tau-})(1-\phi_B)K_{\tau-} + M_{B,\tau-}.
\end{aligned}$$

Therefore, $N_{z,\tau} = N_{z,\tau-}$ for all agents z. As a result, $\eta_\tau - \eta_{\tau-} = \alpha_\tau - \alpha_{\tau-} = 0$. $\square$

Proof of Proposition 2.6. Lemma 2.4 allows usage of the expressions of the expressions in Proposition 2.3 regardless of the type of shock being considered. Lemma 2.5 then allows (α, η) to be held fixed when computing impact responses. Indeed, given the shock is small, the following marginal calculations are sufficient.

Thus, for κ, it suffices to note that $d\tilde{\kappa}/d\Delta_A = 2\tilde{\kappa}\phi_A^2(1-\Delta_A)\hat{\sigma}_A^2\eta^{-1} > 0$ (so the response of $\kappa \in (0,1)$ is strict). For μ^η, compute

$$\frac{d\mu^\eta}{d\Delta_A} = 2\eta(1-\eta)\Big(\underbrace{-\frac{\hat{\pi}^2_{F\to A}}{1-\Delta_A}}_{:=D_1} + \underbrace{\frac{d\log\kappa}{d\Delta_A}\frac{1}{1-\kappa}\Big[\hat{\pi}^2_{F\to A} - \alpha\hat{\pi}^2_A - \frac{\kappa}{\eta(1-\eta)}\mu^\eta\Big]}_{:=D_2} \Big).$$

Note that $D_1 < 0$. Under either of assumptions (i) or (ii), $D_2 \approx 0$. Indeed, given the assumption that the economy is near enough to steady-state, we can ignore the μ^η term. The nearness to steady-state also allows us to re-write $\hat{\pi}^2_{F\to A} - \alpha\hat{\pi}^2_A \approx (1-\alpha)\hat{\pi}^2_{F\to A} - \alpha\hat{\pi}^2_{F\to B}$. Under near-symmetry (i) this term is approximately zero. Under $|\log(\kappa_t/\kappa_{t-})|$ not too large (ii), $d\log\kappa/d\Delta_A$ must be nearly zero. In either case, we have $d\mu^\eta/d\Delta_A < 0$. $\square$

C.3 Equilibrium for Section 3

First, with the presence of aggregate risk, distressed investors, and the leverage constraint, we re-write the optimization problems of all agents. As the first part of Lemma C.2 below shows, the OLG assumptions simply add the death rate δ to the subjective discount rates of agents, so we simply reinterpret agents' discount rates in (1).

The insider of sector $z \in \{A, B\}$ solves

$$\max_{n_i^z, c_i^z, k_i^z, \theta_i^z} \mathcal{U}^z_{i,t}, \quad z \in \{A, B\}, \tag{60}$$

subject to

$$dn^z_{i,t} = \underbrace{(n^z_{i,t}r_t - c^z_{i,t})dt}_{\text{consumption-savings}} + \underbrace{n^z_{i,t}\theta^z_{i,t}\cdot(\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}} + \underbrace{k^z_{i,t}(dR^z_{i,t} - r_t dt)}_{\text{capital ownership}} - \underbrace{\phi_z k^z_{i,t}(d\tilde{R}^z_{i,t} - r_t dt)}_{\text{outside funding}}, \quad z \in \{A, B\}. \tag{61}$$

and $n^z_{i,t} \geq 0$, $k^z_{i,t} \geq 0$, $\theta^z_{i,t} \equiv 0$ (this constraint captures insiders' inability to trade Arrow claims on aggregate shocks).

On the other hand, the hedging portfolios of financiers and distressed investors (θ^F_i, θ^D_i) are unconstrained. Financiers solve

$$\max_{n_i^F, c_i^F, \lambda^A_{F,i}, \lambda^B_{F,i}, \theta_i^F} \mathcal{U}^F_{i,t} \tag{62}$$

subject to

$$dn^F_{i,t} = \underbrace{(n^F_{i,t}r_t - c^F_{i,t})dt}_{\text{consumption-savings}} + \underbrace{n^F_{i,t}\theta^F_{i,t}\cdot(\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}$$
$$+ \underbrace{\lambda^A_{F,i,t}n^F_{i,t}\Delta_A^{-1}\int_i^{i+\Delta_A}(d\tilde{R}^A_{j,t} - r_t dt)dj}_{\text{funding portfolio (sector } A)} + \underbrace{\lambda^B_{F,i,t}n^F_{i,t}\Delta_B^{-1}\int_i^{i+\Delta_B}(d\tilde{R}^B_{j,t} - r_t dt)dj}_{\text{funding portfolio (sector } B)} \tag{63}$$

and $n^F_{i,t} \geq 0$, $\lambda^A_{F,i,t} \geq 0$, $\lambda^B_{F,i,t} \geq 0$, and $\lambda^A_{F,i,t} + \lambda^B_{F,i,t} \leq \bar{\lambda}$. The leverage constraint is now an additional portfolio constraint. Distressed investors solve a similar problem, but without the leverage constraint, and with the additional cost of investment χ:

$$\max_{n^D_i, c^D_i, \lambda^A_{D,i}, \lambda^B_{D,i}, \theta^D_i} \mathcal{U}^D_{i,t} \tag{64}$$

subject to

$$dn^D_{i,t} = \underbrace{(n^D_{i,t}r_t - c^D_{i,t})dt}_{\text{consumption-savings}} + \underbrace{n^D_{i,t}\theta^D_{i,t}\cdot(\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}$$
$$+ \underbrace{\lambda^A_{D,i,t}n^D_{i,t}\Delta_A^{-1}\int_i^{i+\Delta_A}(d\tilde{R}^A_{j,t} - (r_t+\chi) dt)dj}_{\text{funding portfolio (sector } A)} + \underbrace{\lambda^B_{D,i,t}n^D_{i,t}\Delta_B^{-1}\int_i^{i+\Delta_B}(d\tilde{R}^B_{j,t} - (r_t+\chi) dt)dj}_{\text{funding portfolio (sector } B)} \tag{65}$$

and $n^D_{i,t} \geq 0$, $\lambda^A_{D,i,t} \geq 0$, $\lambda^B_{D,i,t} \geq 0$.

We now provide a definition of equilibrium, analogous to Definition 1 in Section 2.

Definition 5. *For the model of Section 3, an* equilibrium *consists of price and allocation processes, adapted to the shocks* $\{(W^A_{i,t}, W^B_{i,t}) : i \in [0,1], t \geq 0\}$ *and* $\{(Z^A_t, Z^B_t) : t \geq 0\}$*, such that all agents solve their optimization problems and all markets clear. Prices consist of the interest rate* r_t*, spreads* $(s^A_{i,t}, s^B_{i,t})$*, and the aggregate risk price vector* π_t*. Allocations consist of capital and equity holdings* $(k^A_{i,t}, k^B_{i,t}, \lambda^A_{F,i,t}, \lambda^B_{F,i,t}, \lambda^A_{D,i,t}, \lambda^B_{D,i,t})$*, investment in Arrow claims on aggregate shocks* $(\theta^A_{i,t}, \theta^B_{i,t}, \theta^F_{i,t}, \theta^D_{i,t})$*, and consumption choices* $(c^A_{i,t}, c^B_{i,t}, c^F_{i,t}, c^D_{i,t})$*. A* symmetric equilibrium *is an equilibrium in which all objects are independent of i for each t. The market-clearing conditions are as follows.*

- Goods:

$$\int_0^1 [G_A k^A_{i,t} + G_B k^B_{i,t}]di = \int_0^1 [c^A_{i,t} + c^B_{i,t} + c^F_{i,t} + c^D_{i,t}]di + \frac{1}{dt}\int_0^1 [dI^A_{i,t} + dI^B_{i,t}]di + \chi \int_0^1 n^D_{i,t}(\lambda^A_{D,i,t} + \lambda^B_{D,i,t})di.$$

- Funding:

$$\int_{i-\Delta_z}^{i} \Delta_z^{-1}[\lambda^z_{F,j,t}n^F_{i,t} + \lambda^z_{D,j,t}n^D_{i,t}]dj = \phi_z k^z_{i,t}, \quad \forall i \in [0,1], \quad z \in \{A, B\}.$$

- Aggregate risk (Arrow securities on Z^A and Z^B):

$$\int_0^1 [\theta^A_{i,t}n^A_{i,t} + \theta^B_{i,t}n^B_{i,t} + \theta^F_{i,t}n^F_{i,t} + \theta^D_{i,t}n^D_{i,t}]di = 0.$$

- Bond:

$$\int_0^1 [n^A_{i,t} + n^B_{i,t} + n^F_{i,t} + n^D_{i,t}]di = \int_0^1 [k^A_{i,t} + k^B_{i,t}]di.$$

Lemma C.2. *Consider the OLG framework of Section 3. Equilibrium holds with subjective discount rate ρ_z for agent $z \in \{A, B, F, D\}$ replaced by $\tilde{\rho}_z := \rho_z + \delta$, and with μ^α, μ^η, and μ^x augmented with $\delta((\nu_A + \nu_B)^{-1}\nu_A - \alpha)$, $\delta(\nu_F + \nu_D - \eta)$, and $\delta((\nu_F + \nu_D)^{-1}\nu_F - x)$ i.e., replaced by*

$$\begin{aligned}
\mu^\alpha &= \mu_0^\alpha + \delta((\nu_A + \nu_B)^{-1}\nu_A - \alpha) \\
\mu^\eta &= \mu_0^\eta + \delta(\nu_F + \nu_D - \eta) \\
\mu^x &= \mu_0^x + \delta((\nu_F + \nu_D)^{-1}\nu_F - x),
\end{aligned}$$

where $\mu_0^\alpha, \mu_0^\eta, \mu_0^x$ come from the economy with $\delta = 0$.

Proof of Lemma C.2. A proof of the fact that subjective discount rates in recursive preferences are simply augmented by the Poisson death rate can be found in the appendix of Gârleanu and Panageas (2015). OLG adds the following terms to the dynamics of aggregate net worth:

$$dN_{z,t} = \ldots - \delta N_{z,t}dt + \nu_z \delta K_t dt.$$

Applying Itô's formula to the wealth shares (α, η, x) yields the result on the state drifts. □

Proposition C.3 (Equilibrium with Distress)**.** *Let Assumptions 1 and 2 hold, and augment financiers' problem with constraint (17). Let $(\hat{\pi}_A, \hat{\pi}_B)$ be insiders' idiosyncratic risk prices, defined in (12), and let $(\hat{\pi}_{F\to A}, \hat{\pi}_{F\to B})$ be financiers' idiosyncratic risk prices, defined by (13) with $x\eta$ in place of η. In a symmetric equilibrium, (κ, ζ) solve a nonlinear system given by (19) and equation (69). Equilibrium spreads (s_A, s_B) are given by*

$$\begin{aligned}
s_z - \sigma_z \cdot \pi = x\Big[&(1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z} + \zeta - \big(\zeta - \chi - \frac{x}{1-x}(1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}\big)^+\Big] \\
&+ (1-x)\Big[\chi - \big(\chi - \zeta - (1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}\big)^+\Big], \quad z \in \{A, B\},
\end{aligned} \tag{66}$$

where $\pi = \eta^{-1}[\kappa\phi_A\sigma_A + (1-\kappa)\phi_B\sigma_B]$ is the traded aggregate risk price.

Proof of Proposition C.3. In the proof below, I restrict attention to symmetric equilibria, in which all equilibrium objects are independent of i. To simplify notation, I drop all i subscripts when the meaning is clear. Within the class of symmetric equilibria, I solve for the equilibrium objects in two steps. In the first step, I assume (κ_t, ζ_t) are known and use them to solve for all other objects. In the second step, I solve for (κ_t, ζ_t) via a system of nonlinear equations. In this proof, I treat the case $\delta = 0$. The general case with $\delta > 0$ is accounted for by Lemma C.2 above.

Step 1. Solving for equilibrium given (κ, ζ).

Using the optimal consumption decisions from Appendix B.4, we can write the goods market clearing condition as (21). Thus, the aggregate investment rate ι is solved, given κ and λ_D^A, λ_D^B.

Next, apply optimal portfolio choice as special cases of the general formulas derived in Appendix B.4. Define $\pi_A := \hat{\pi}_A\sigma_A/\hat{\sigma}_A$ and $\pi_B := \hat{\pi}_B\sigma_B/\hat{\sigma}_B$ where $\hat{\pi}_A = \kappa(1-\phi_A)\hat{\sigma}_A/(1-\eta)\alpha$ and $\hat{\pi}_B = (1-\kappa)(1-\phi_B)\hat{\sigma}_B/(1-\eta)(1-\alpha)$ are defined by (12). With this notation, insiders' optimal capital portfolio choice can be written

$$(1-\phi_A)[\hat{\sigma}_A\hat{\pi}_A + \sigma_A \cdot \pi_A] \geq G_A - r - \phi_A s_A \quad \text{with equality when } \kappa > 0 \tag{67}$$

$$(1-\phi_B)[\hat{\sigma}_B\hat{\pi}_B + \sigma_B \cdot \pi_B] \geq G_B - r - \phi_B s_B \quad \text{with equality when } \kappa < 1. \tag{68}$$

By taking the difference between (67) and (68), and noting that at least one of these is always an equality, we obtain $0 = \min(1-\kappa, \max(-\kappa, H))$, or

$$\begin{aligned}
&0 = \min\{1-\kappa,\, H^+\} - \min\{\kappa,\, (-H)^+\} \\
&H := G_A - G_B - \phi_A s_A + \phi_B s_B - (1-\phi_A)[\sigma_A \cdot \pi_A + \hat{\sigma}_A\hat{\pi}_A] + (1-\phi_B)[\sigma_B \cdot \pi_B + \hat{\sigma}_B\hat{\pi}_B].
\end{aligned} \tag{69}$$

The interest rate r is found by multiplying (67)-(68) by κ and $1-\kappa$, respectively, and summing.

Exposures to aggregate risk dZ_t are determined as follows. Note that π_A, π_B represent insiders' exposures to aggregate risk, by its definition. For financiers and distressed investors, who can trade aggregate risk without constraints, their optimal exposure is equal to the aggregate risk price vector π. Thus, $\theta_F + \lambda_F^A\sigma_A + \lambda_F^B\sigma_B = \theta_D +$

$\lambda_D^A\sigma_A + \lambda_D^B\sigma_B = \pi$. In equilibrium, the aggregate risk price vector is determined by applying aggregate risk market clearing, which can be restated as

$$\eta[x\pi + (1-x)\pi] + (1-\eta)[\alpha\pi_A + (1-\alpha)\pi_B] = \kappa\sigma_A + (1-\kappa)\sigma_B.$$

Substituting π_A, π_B, we solve for $\pi = \eta^{-1}[\kappa\phi_A\sigma_A + (1-\kappa)\phi_B\sigma_B]$.

Next, we determine spreads by using funding market clearing, which aggregates to

$$x\lambda_F^A + (1-x)\lambda_D^A = \frac{\kappa\phi_A}{\eta} \quad \text{and} \quad x\lambda_F^B + (1-x)\lambda_D^B = \frac{(1-\kappa)\phi_B}{\eta}.$$

By the analysis of Appendix B.4, (18) and (20) represent optimal lending positions. Substituting into funding market clearing, we have

$$x(s_A - \sigma_A \cdot \pi - \zeta)^+ + (1-x)(s_A - \sigma_A \cdot \pi - \chi)^+ = \frac{\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2}{\eta} := x(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A},$$

and symmetrically for sector B. Note that $s_A - \sigma_A \cdot \pi > \min(\zeta, \chi)$ is required for this equation to hold. The mutually-exclusive, completely exhaustive cases are as follows. If $s_A - \sigma_A \cdot \pi \geq \max(\zeta, \chi)$, then

$$s_A - \sigma_A \cdot \pi = x\zeta + (1-x)\chi + x(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}.$$

Thus, this case obtains when $\zeta - \chi - \frac{x}{1-x}(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \leq 0$ and $\chi - \zeta - (1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \leq 0$, which implies (66) holds. If $\chi \geq s_A - \sigma_A \cdot \pi > \zeta$, then

$$s_A - \sigma_A \cdot \pi = \zeta + (1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}.$$

Thus, this case obtains when $\chi - \zeta - (1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \geq 0$, which implies (66) holds. If $\zeta \geq s_A - \sigma_A \cdot \pi > \chi$, then

$$s_A - \sigma_A \cdot \pi = \chi + \frac{x}{1-x}(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}.$$

Thus, this case obtains when $\zeta - \chi - \frac{x}{1-x}(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \geq 0$, which implies (66) holds. Combining these results, equation (66) holds in all cases. An identical analysis holds for sector B.

Finally, we determine state variable dynamics. Define the following expressions:

$$\begin{aligned}
\Pi_F &:= \theta_F \cdot \pi + \lambda_F^A s_A + \lambda_F^B s_B \\
\Pi_D &:= \theta_D \cdot \pi + \lambda_D^A(s_A - \chi) + \lambda_D^B(s_B - \chi) \\
\Pi_A &:= \theta_A \cdot \pi + (1-\eta)^{-1}\alpha^{-1}\kappa(G_A - r - \phi_A s_A) \\
\Pi_B &:= \theta_B \cdot \pi + (1-\eta)^{-1}(1-\alpha)^{-1}(1-\kappa)(G_B - r - \phi_B s_B).
\end{aligned}$$

Note that $K_A/N_A = \kappa/(1-\eta)\alpha$ and $K_B/N_B = (1-\kappa)/(1-\eta)(1-\alpha)$ by bond market clearing. Thus, aggregating agents' net worth evolutions, which eliminates any contributions from $\{dW_{i,t}\}_{i\in[0,1]}$ due to Lemma 2.2, and substituting these expressions for $\Pi_F, \Pi_D, \Pi_A, \Pi_B$, we obtain

$$\frac{dN_F}{N_F} = \big[r - \rho_F + \Pi_F\big]dt + \big[\theta_F + \lambda_F^A\sigma_A + \lambda_F^B\sigma_B\big]dZ_t \tag{70}$$

$$\frac{dN_D}{N_D} = \big[r - \rho + \Pi_D\big]dt + \big[\theta_D + \lambda_D^A\sigma_A + \lambda_D^B\sigma_B\big]dZ_t \tag{71}$$

$$\frac{dN_A}{N_A} = \big[r - \rho + \Pi_A\big]dt + \big[\theta_A + (1-\eta)^{-1}\alpha^{-1}\kappa\sigma_A\big]dZ_t \tag{72}$$

$$\frac{dN_B}{N_B} = \big[r - \rho + \Pi_B\big]dt + \big[\theta_B + (1-\eta)^{-1}(1-\alpha)^{-1}(1-\kappa)\sigma_B\big]dZ_t. \tag{73}$$

Using expressions (67)-(68), we have shown that $\Pi_F, \Pi_D, \Pi_A, \Pi_B$ are equivalent to

$$\Pi_A = \hat{\pi}_A^2 + \|\pi_A\|^2 \quad \text{and} \quad \Pi_B := \hat{\pi}_B^2 + \|\pi_B\|^2 \tag{74}$$

$$\Pi_F = \lambda_F^A(s_A - \sigma_A \cdot \pi) + \lambda_F^B(s_B - \sigma_B \cdot \pi) + \|\pi\|^2 \tag{75}$$

$$\Pi_D = \lambda_D^A(s_A - \sigma_A \cdot \pi - \chi) + \lambda_D^B(s_B - \sigma_B \cdot \pi - \chi) + \|\pi\|^2. \tag{76}$$

Furthermore, using the net worth evolutions in (70)-(72) and substituting the optimal aggregate risk exposures derived above, then applying Itô's formula to the definitions of (α,η,x), we obtain the state variable evolutions

$$\begin{aligned}
\mu^{\alpha} &= \alpha(1-\alpha)[\Pi_A-\Pi_B]-(\alpha\pi_A+(1-\alpha)\pi_B)\cdot\sigma^{\alpha}+\delta((\nu_A+\nu_B)^{-1}\nu_A-\alpha) && (77)\\
\sigma^{\alpha} &= \alpha(1-\alpha)[\pi_A-\pi_B], && (78)\\
\mu^{\eta} &= \eta(1-\eta)[x(\rho-\rho_F)+x\Pi_F+(1-x)\Pi_D-\alpha\Pi_A-(1-\alpha)\Pi_B] && (79)\\
&\quad -(\eta\pi+(1-\eta)(\alpha\pi_A+(1-\alpha)\pi_B))\cdot\sigma^{\eta}+\delta(\nu_F+\nu_D-\eta)\\
\sigma^{\eta} &= \eta(1-\eta)[\pi-\alpha\pi_A-(1-\alpha)\pi_B], && (80)\\
\mu^{x} &= x(1-x)[\rho-\rho_F+\Pi_F-\Pi_D]+\delta((\nu_F+\nu_D)^{-1}\nu_F-x) && (81)\\
\sigma^{x} &= 0. && (82)
\end{aligned}$$

Step 2. Solving for (κ,ζ).

Substituting π, (18), (66) into (19), we obtain a single equation in (κ,ζ). Substituting π,π_A,π_B, (66) into (69), we obtain a second equation in (κ,ζ). A solution exists, as demonstrated by Proposition C.4. □

Proposition C.4 below states the analytical solution to this equilibrium by explicitly solving equations (19) and (69). An explicit solution is possible because the nonlinearity of this system is induced solely by the various portfolio constraints (i.e., leverage, shorting constraints). Such constraints bind on endogenous subsets of the state space, which I solve for analytically.

Proposition C.4. *In the equilibrium of Proposition C.3, the solution (κ,ζ) to (19) and (69) is determined as follows. Consider the state space $\Omega := [0,1]^3$ for (α,η,x). First define the following objects on Ω:*

$$\Gamma_A = \begin{cases} 1, & \text{if insiders may frictionlessly trade aggregate risk (unconstrained } \theta_A,\theta_B)\\ \frac{\phi_A^2}{\eta}+\frac{(1-\phi_A)^2}{\alpha(1-\eta)}, & \text{if insiders may not trade aggregate risk } (\theta_A=\theta_B\equiv 0),\end{cases}$$

$$\Gamma_B = \begin{cases} 1, & \text{if insiders may frictionlessly trade aggregate risk (unconstrained } \theta_A,\theta_B)\\ \frac{\phi_B^2}{\eta}+\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)}, & \text{if insiders may not trade aggregate risk } (\theta_A=\theta_B\equiv 0),\end{cases}$$

and

$$\begin{aligned}
\Sigma_A &:= \frac{(1-\Delta_A)^2\hat{\sigma}_A^2}{\eta(1-x)}\\
\Sigma_B &:= \frac{(1-\Delta_B)^2\hat{\sigma}_B^2}{\eta(1-x)}\\
M_A &:= \Gamma_A\|\sigma_A\|^2+\Big[\frac{(1-\phi_A)^2}{\alpha(1-\eta)}+\frac{\phi_A^2(1-\Delta_A)^2}{\eta}\Big]\hat{\sigma}_A^2\\
M_B &:= \Gamma_B\|\sigma_B\|^2+\Big[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)}+\frac{\phi_B^2(1-\Delta_B)^2}{\eta}\Big]\hat{\sigma}_B^2.
\end{aligned}$$

Next, for each $(\alpha,\eta,x)\in\Omega$, define the following function mapping $[0,1]\mapsto\mathbb{R}$:

$$\begin{aligned}
\tilde{\lambda}(\kappa) &:= \mathbf{1}_{\{\Delta_A=1\}}\frac{\kappa\phi_A}{x\eta}+\mathbf{1}_{\{\Delta_A<1\}}\Big\{x\frac{\kappa\phi_A}{x\eta}+(1-x)\frac{\chi}{(1-\Delta_A)^2\hat{\sigma}_A^2}-(1-x)\Big[\frac{\chi}{(1-\Delta_A)^2\hat{\sigma}_A^2}-\frac{\kappa\phi_A}{x\eta}\Big]^+\Big\}\\
&\quad+\mathbf{1}_{\{\Delta_B=1\}}\frac{(1-\kappa)\phi_B}{x\eta}+\mathbf{1}_{\{\Delta_B<1\}}\Big\{x\frac{(1-\kappa)\phi_B}{x\eta}+(1-x)\frac{\chi}{(1-\Delta_B)^2\hat{\sigma}_B^2}-(1-x)\Big[\frac{\chi}{(1-\Delta_B)^2\hat{\sigma}_B^2}-\frac{(1-\kappa)\phi_B}{x\eta}\Big]^+\Big\}.
\end{aligned}$$

Also define the following functions mapping $\mathbb{R} \mapsto \mathbb{R}$:

$$
\begin{aligned}
\tilde{\kappa}(\zeta) &:= \frac{G_A - G_B + M_B - (\phi_A - \phi_B)(x\zeta + (1-x)\chi)}{M_A + M_B} \\
\tilde{\kappa}_{F\not\to A}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A\chi + \phi_B(x\zeta + (1-x)\chi)}{M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2} \\
\tilde{\kappa}_{F\not\to B}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A(x\zeta + (1-x)\chi) + \phi_B\chi + x(1-x)^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}{M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2} \\
\tilde{\kappa}_{D\not\to A}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A\zeta + \phi_B(x\zeta + (1-x)\chi)}{M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2} \\
\tilde{\kappa}_{D\not\to B}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A(x\zeta + (1-x)\chi) + \phi_B\zeta + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}{M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2} \\
\tilde{\kappa}_{D\not\to A,B}(\zeta) &:= \frac{G_A - G_B + M_B + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}{M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2 + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}[1 - \mathbf{1}_{\{\zeta>0,\phi_A\neq\phi_B\}}] \\
&\quad + \frac{x\eta\bar{\lambda} - \phi_B}{\phi_A - \phi_B}\mathbf{1}_{\{\zeta>0,\phi_A\neq\phi_B\}}.
\end{aligned}
$$

Using these functions, for each $(\alpha, \eta, x) \in \Omega$, define the following objects:

$$
\begin{aligned}
\tilde{\kappa}_0^* &:= [\tilde{\kappa}(0)]^+ \wedge 1 \\
\tilde{\kappa}_{0,D\not\to A}^* &:= [\tilde{\kappa}_{D\not\to A}(0)]^+ \wedge 1 \\
\tilde{\kappa}_{0,D\not\to B}^* &:= [\tilde{\kappa}_{D\not\to B}(0)]^+ \wedge 1 \\
\tilde{\kappa}_{0,D\not\to A,B}^* &:= [\tilde{\kappa}_{D\not\to A,B}(0)]^+ \wedge 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\zeta}^* &:= \frac{(\Sigma_A^{-1} + \Sigma_B^{-1})\chi + \tilde{\kappa}(0)\phi_A + (1-\tilde{\kappa}(0))\phi_B - \eta\bar{\lambda}}{\Sigma_A^{-1} + \Sigma_B^{-1} + x[M_A + M_B]^{-1}(\phi_A - \phi_B)^2} \\
\tilde{\zeta}_{F\not\to A}^* &:= \frac{\Sigma_B^{-1}\chi + (1 - \tilde{\kappa}_{F\not\to A}(0))\phi_B - \eta\bar{\lambda}}{\Sigma_B^{-1} + x[M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2]^{-1}\phi_B^2} \\
\tilde{\zeta}_{F\not\to B}^* &:= \frac{\Sigma_A^{-1}\chi + \tilde{\kappa}_{F\not\to B}(0)\phi_A - \eta\bar{\lambda}}{\Sigma_A^{-1} + x[M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2]^{-1}\phi_A^2} \\
\tilde{\zeta}_{D\not\to A}^* &:= \frac{\Sigma_B^{-1}\chi + \tilde{\kappa}_{D\not\to A}(0)x^{-1}\phi_A + (1 - \tilde{\kappa}_{D\not\to A}(0))\phi_B - \eta\bar{\lambda}}{\Sigma_B^{-1} + x[M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2]^{-1}(x^{-1}\phi_A - \phi_B)^2} \\
\tilde{\zeta}_{D\not\to B}^* &:= \frac{\Sigma_A^{-1}\chi + \tilde{\kappa}_{D\not\to B}(0)\phi_A + (1 - \tilde{\kappa}_{D\not\to B}(0))x^{-1}\phi_B - \eta\bar{\lambda}}{\Sigma_A^{-1} + x[M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2]^{-1}(\phi_A - x^{-1}\phi_B)^2} \\
\tilde{\zeta}_{D\not\to A,B}^* &:= \frac{\mathbf{1}_{\{\phi_A\neq\phi_B\}}}{(\phi_A - \phi_B)^2}\Big[(G_A - G_B)(\phi_A - \phi_B) + \Big(M_B + \frac{1-x}{x}\frac{\phi_B^2(1-\Delta_B)^2}{\eta}\hat{\sigma}_B^2\Big)(\phi_A - x\eta\bar{\lambda}) \\
&\quad + \Big(M_A + \frac{1-x}{x}\frac{\phi_A^2(1-\Delta_A)^2}{\eta}\hat{\sigma}_A^2\Big)(\phi_B - x\eta\bar{\lambda})\Big],
\end{aligned}
$$

and

$$\begin{aligned}
\tilde{\kappa}^*_\zeta &:= \tilde{\kappa}(\tilde{\zeta}^*) \\
\tilde{\kappa}^*_{\zeta,F\not\to A} &:= \tilde{\kappa}_{F\not\to A}(\tilde{\zeta}^*_{F\not\to A}) \\
\tilde{\kappa}^*_{\zeta,F\not\to B} &:= \tilde{\kappa}_{F\not\to B}(\tilde{\zeta}^*_{F\not\to B}) \\
\tilde{\kappa}^*_{\zeta,D\not\to A} &:= \tilde{\kappa}_{D\not\to A}(\tilde{\zeta}^*_{D\not\to A}) \\
\tilde{\kappa}^*_{\zeta,D\not\to B} &:= \tilde{\kappa}_{D\not\to B}(\tilde{\zeta}^*_{D\not\to B}) \\
\tilde{\kappa}^*_{\zeta,D\not\to A,B} &:= \tilde{\kappa}_{D\not\to A,B}(\tilde{\zeta}^*_{D\not\to A,B}),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\zeta}^{\kappa=1} &:= \chi + (\Sigma_A^{-1} + \Sigma_B^{-1})^{-1}[\phi_A - \eta\bar{\lambda}] \\
\tilde{\zeta}^{\kappa=1}_{D\not\to A} &:= \chi + \Sigma_B[x^{-1}\phi_A - \eta\bar{\lambda}] \\
\tilde{\zeta}^{\kappa=0} &:= \chi + (\Sigma_A^{-1} + \Sigma_B^{-1})^{-1}[\phi_B - \eta\bar{\lambda}] \\
\tilde{\zeta}^{\kappa=0}_{D\not\to B} &:= \chi + \Sigma_A[x^{-1}\phi_B - \eta\bar{\lambda}].
\end{aligned}$$

Finally, for each $(\alpha, \eta, x) \in \Omega$*, define the following functions mapping* $[0,1] \times \mathbb{R} \mapsto \mathbb{R}$*:*

$$\begin{aligned}
\tilde{d}_A(\kappa,\zeta) &:= (x\eta)^{-1}\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2 + \zeta - \chi \\
\tilde{d}_B(\kappa,\zeta) &:= (x\eta)^{-1}(1-\kappa)\phi_B(1-\Delta_B)^2\hat{\sigma}_B^2 + \zeta - \chi \\
\tilde{f}_A(\kappa,\zeta) &:= ((1-x)\eta)^{-1}\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2 - \zeta + \chi \\
\tilde{f}_B(\kappa,\zeta) &:= ((1-x)\eta)^{-1}(1-\kappa)\phi_B(1-\Delta_B)^2\hat{\sigma}_B^2 - \zeta + \chi.
\end{aligned}$$

Using all the definitions above, construct the following subsets of Ω*:*

$$\begin{aligned}
\Omega_0 &:= \left\{(\alpha,\eta,x) \in \Omega : \tilde{d}_A(\tilde{\kappa}^*_0, 0) \wedge \tilde{d}_B(\tilde{\kappa}^*_0, 0) \geq 0,\ \tilde{\lambda}(\tilde{\kappa}^*_0) \leq \bar{\lambda}\right\} \\
\Omega_{0,D\not\to A} &:= \left\{(\alpha,\eta,x) \in \Omega : \tilde{d}_A(\tilde{\kappa}^*_{0,D\not\to A}, 0) < 0 \leq \tilde{d}_B(\tilde{\kappa}^*_{0,D\not\to A}, 0),\ \tilde{\lambda}(\tilde{\kappa}^*_{0,D\not\to A}) \leq \bar{\lambda}\right\} \\
\Omega_{0,D\not\to B} &:= \left\{(\alpha,\eta,x) \in \Omega : \tilde{d}_B(\tilde{\kappa}^*_{0,D\not\to B}, 0) < 0 \leq \tilde{d}_A(\tilde{\kappa}^*_{0,D\not\to B}, 0),\ \tilde{\lambda}(\tilde{\kappa}^*_{0,D\not\to B}) \leq \bar{\lambda}\right\} \\
\Omega_{0,D\not\to A,B} &:= \left\{(\alpha,\eta,x) \in \Omega : \tilde{d}_A(\tilde{\kappa}^*_{0,D\not\to A,B}, 0) \vee \tilde{d}_B(\tilde{\kappa}^*_{0,D\not\to A,B}, 0) < 0,\ \tilde{\lambda}(\tilde{\kappa}^*_{0,D\not\to A,B}) \leq \bar{\lambda}\right\} \\
\Omega_1 &:= \Omega \backslash (\Omega_0 \cup \Omega_{0,D\not\to A} \cup \Omega_{0,D\not\to B} \cup \Omega_{0,D\not\to A,B}),
\end{aligned}$$

and

$$\begin{aligned}
\Omega_\zeta := \{(\alpha,\eta,x) \in \Omega_1 :\ & \tilde{d}_A(\tilde{\kappa}^*_\zeta, \tilde{\zeta}^*) \wedge \tilde{d}_B(\tilde{\kappa}^*_\zeta, \tilde{\zeta}^*) \geq 0, \\
& \tilde{f}_A(\tilde{\kappa}^*_\zeta, \tilde{\zeta}^*) \wedge \tilde{f}_B(\tilde{\kappa}^*_\zeta, \tilde{\zeta}^*) \geq 0,\ 0 \leq \tilde{\kappa}^*_\zeta \leq 1,\ \tilde{\zeta}^* > 0\} \\
\Omega_{\zeta,D\not\to A} := \{(\alpha,\eta,x) \in \Omega_1 :\ & \tilde{d}_A(\tilde{\kappa}^*_{\zeta,D\not\to A}, \tilde{\zeta}^*_{D\not\to A}) < 0 \leq \tilde{d}_B(\tilde{\kappa}^*_{\zeta,D\not\to A}, \tilde{\zeta}^*_{D\not\to A}), \\
& \tilde{f}_A(\tilde{\kappa}^*_{\zeta,D\not\to A}, \tilde{\zeta}^*_{D\not\to A}) \wedge \tilde{f}_B(\tilde{\kappa}^*_{\zeta,D\not\to A}, \tilde{\zeta}^*_{D\not\to A}) \geq 0,\ 0 \leq \tilde{\kappa}^*_{\zeta,D\not\to A} \leq 1,\ \tilde{\zeta}^*_{D\not\to A} > 0\} \\
\Omega_{\zeta,D\not\to B} := \{(\alpha,\eta,x) \in \Omega_1 :\ & \tilde{d}_B(\tilde{\kappa}^*_{\zeta,D\not\to B}, \tilde{\zeta}^*_{D\not\to B}) < 0 \leq \tilde{d}_A(\tilde{\kappa}^*_{\zeta,D\not\to B}, \tilde{\zeta}^*_{D\not\to B}), \\
& \tilde{f}_A(\tilde{\kappa}^*_{\zeta,D\not\to B}, \tilde{\zeta}^*_{D\not\to B}) \wedge \tilde{f}_B(\tilde{\kappa}^*_{\zeta,D\not\to B}, \tilde{\zeta}^*_{D\not\to B}) \geq 0,\ 0 \leq \tilde{\kappa}^*_{\zeta,D\not\to B} \leq 1,\ \tilde{\zeta}^*_{D\not\to B} > 0\} \\
\Omega_{\zeta,D\not\to A,B} := \{(\alpha,\eta,x) \in \Omega_1 :\ & \tilde{d}_A(\tilde{\kappa}^*_{\zeta,D\not\to A,B}, \tilde{\zeta}^*_{D\not\to A,B}) \vee \tilde{d}_B(\tilde{\kappa}^*_{\zeta,D\not\to A,B}, \tilde{\zeta}^*_{D\not\to A,B}) < 0, \\
& \tilde{f}_A(\tilde{\kappa}^*_{\zeta,D\not\to A,B}, \tilde{\zeta}^*_{D\not\to A,B}) \wedge \tilde{f}_B(\tilde{\kappa}^*_{\zeta,D\not\to A,B}, \tilde{\zeta}^*_{D\not\to A,B}) \geq 0,\ 0 \leq \tilde{\kappa}^*_{\zeta,D\not\to A,B} \leq 1,\ \tilde{\zeta}^*_{D\not\to A,B} > 0\} \\
\Omega_{\zeta,F\not\to A} := \{(\alpha,\eta,x) \in \Omega_1 :\ & \tilde{f}_A(\tilde{\kappa}^*_{\zeta,F\not\to A}, \tilde{\zeta}^*_{F\not\to A}) < 0 \leq \tilde{f}_B(\tilde{\kappa}^*_{\zeta,F\not\to A}, \tilde{\zeta}^*_{F\not\to A}), \\
& \tilde{d}_A(\tilde{\kappa}^*_{\zeta,F\not\to A}, \tilde{\zeta}^*_{F\not\to A}) \wedge \tilde{d}_B(\tilde{\kappa}^*_{\zeta,F\not\to A}, \tilde{\zeta}^*_{F\not\to A}) \geq 0,\ 0 \leq \tilde{\kappa}^*_{\zeta,F\not\to A} \leq 1,\ \tilde{\zeta}^*_{F\not\to A} > 0\} \\
\Omega_{\zeta,F\not\to B} := \{(\alpha,\eta,x) \in \Omega_1 :\ & \tilde{f}_B(\tilde{\kappa}^*_{\zeta,F\not\to B}, \tilde{\zeta}^*_{F\not\to B}) < 0 \leq \tilde{f}_A(\tilde{\kappa}^*_{\zeta,F\not\to B}, \tilde{\zeta}^*_{F\not\to B}), \\
& \tilde{d}_A(\tilde{\kappa}^*_{\zeta,F\not\to B}, \tilde{\zeta}^*_{F\not\to B}) \wedge \tilde{d}_B(\tilde{\kappa}^*_{\zeta,F\not\to B}, \tilde{\zeta}^*_{F\not\to B}) \geq 0,\ 0 \leq \tilde{\kappa}^*_{\zeta,F\not\to B} \leq 1,\ \tilde{\zeta}^*_{F\not\to B} > 0\}
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{\zeta}^{\kappa=1} &:= \{(\alpha,\eta,x) \in \Omega_1 : \tilde{d}_A(1,\tilde{\zeta}^{\kappa=1}) \geq 0,\, \tilde{f}_A(1,\tilde{\zeta}^{\kappa=1}) \geq 0,\, \tilde{\kappa}(\tilde{\zeta}^{\kappa=1}) > 1,\, \tilde{\zeta}^{\kappa=1} > 0\} \\
\Omega_{\zeta,D\not\to A}^{\kappa=1} &:= \{(\alpha,\eta,x) \in \Omega_1 : \tilde{d}_A(1,\tilde{\zeta}_{D\not\to A}^{\kappa=1}) < 0,\, \tilde{f}_A(1,\tilde{\zeta}_{D\not\to A}^{\kappa=1}) \geq 0,\, \tilde{\kappa}_{D\not\to A}(\tilde{\zeta}_{D\not\to A}^{\kappa=1}) > 1,\, \tilde{\zeta}_{D\not\to A}^{\kappa=1} > 0\} \\
\Omega_{\zeta}^{\kappa=0} &:= \{(\alpha,\eta,x) \in \Omega_1 : \tilde{d}_B(0,\tilde{\zeta}^{\kappa=0}) \geq 0,\, \tilde{f}_B(0,\tilde{\zeta}^{\kappa=0}) \geq 0,\, \tilde{\kappa}(\tilde{\zeta}^{\kappa=0}) < 0,\, \tilde{\zeta}^{\kappa=0} > 0\} \\
\Omega_{\zeta,D\not\to B}^{\kappa=0} &:= \{(\alpha,\eta,x) \in \Omega_1 : \tilde{d}_B(0,\tilde{\zeta}_{D\not\to B}^{\kappa=0}) < 0,\, \tilde{f}_B(0,\tilde{\zeta}_{D\not\to B}^{\kappa=0}) \geq 0,\, \tilde{\kappa}_{D\not\to B}(\tilde{\zeta}_{D\not\to B}^{\kappa=0}) < 0,\, \tilde{\zeta}_{D\not\to B}^{\kappa=0} > 0\}.
\end{aligned}$$

Then, the solutions for $\kappa : \Omega \mapsto \mathcal{O}$ and $\zeta : \Omega \mapsto \mathcal{O}$, where $\mathcal{O}$ are the finite subsets of $\mathbb{R}_+$, are[41]

$$\kappa \ni \begin{cases}
\tilde{\kappa}_0^*, & on \quad \Omega_0 \\
\tilde{\kappa}_{0,D\not\to A}^*, & on \quad \Omega_{0,D\not\to A} \\
\tilde{\kappa}_{0,D\not\to B}^*, & on \quad \Omega_{0,D\not\to B} \\
\tilde{\kappa}_{0,D\not\to A,B}^*, & on \quad \Omega_{0,D\not\to A,B} \\
\tilde{\kappa}_{\zeta}^*, & on \quad \Omega_{\zeta} \\
\tilde{\kappa}_{\zeta,D\not\to A}^*, & on \quad \Omega_{\zeta,D\not\to A} \\
\tilde{\kappa}_{\zeta,D\not\to B}^*, & on \quad \Omega_{\zeta,D\not\to B} \\
\tilde{\kappa}_{\zeta,D\not\to A,B}^*, & on \quad \Omega_{\zeta,D\not\to A,B} \\
\tilde{\kappa}_{\zeta,F\not\to A}^*, & on \quad \Omega_{\zeta,F\not\to A} \\
\tilde{\kappa}_{\zeta,F\not\to B}^*, & on \quad \Omega_{\zeta,F\not\to B} \\
1, & on \quad \Omega_{\zeta}^{\kappa=1} \cup \Omega_{\zeta,D\not\to A}^{\kappa=1} \\
0, & on \quad \Omega_{\zeta}^{\kappa=0} \cup \Omega_{\zeta,D\not\to B}^{\kappa=0}
\end{cases} \qquad and \qquad \zeta \ni \begin{cases}
\tilde{\zeta}^*, & on \quad \Omega_{\zeta} \\
\tilde{\zeta}_{D\not\to A}^*, & on \quad \Omega_{\zeta,D\not\to A} \\
\tilde{\zeta}_{D\not\to B}^*, & on \quad \Omega_{\zeta,D\not\to B} \\
\tilde{\zeta}_{D\not\to A,B}^*, & on \quad \Omega_{\zeta,D\not\to A,B} \\
\tilde{\zeta}_{F\not\to A}^*, & on \quad \Omega_{\zeta,F\not\to A} \\
\tilde{\zeta}_{F\not\to B}^*, & on \quad \Omega_{\zeta,F\not\to B} \\
\tilde{\zeta}^{\kappa=1}, & on \quad \Omega_{\zeta}^{\kappa=1} \\
\tilde{\zeta}_{D\not\to A}^{\kappa=1}, & on \quad \Omega_{\zeta,D\not\to A}^{\kappa=1} \\
\tilde{\zeta}^{\kappa=0}, & on \quad \Omega_{\zeta}^{\kappa=0} \\
\tilde{\zeta}_{D\not\to B}^{\kappa=0}, & on \quad \Omega_{\zeta,D\not\to B}^{\kappa=0} \\
0, & on \quad \Omega \backslash \Omega_1.
\end{cases}$$

Proof of Proposition C.4. One can substitute these formulas into (19) and (69) to verify that the equations are solved. It remains to show that the union of the regions defined is equal to the entire state space, i.e.,

$$\Omega_0 \cup \Omega_{0,D\not\to A} \cup \Omega_{0,D\not\to B} \cup \Omega_{0,D\not\to A,B} \cup \Omega_1 = \Omega \tag{83}$$

and

$$\Omega_1^* = \Omega_1, \tag{84}$$

where $\Omega_1^* := \Omega_{\zeta} \cup \Omega_{\zeta,D\to A} \cup \Omega_{\zeta,D\to B} \cup \Omega_{\zeta,D\to A,B} \cup \Omega_{\zeta,F\to A} \cup \Omega_{\zeta,F\to B} \cup \Omega_{\zeta}^{\kappa=1} \cup \Omega_{\zeta}^{\kappa=0} \cup \Omega_{\zeta,D\to A}^{\kappa=1} \cup \Omega_{\zeta,D\to B}^{\kappa=0}$. Statement (83) is trivially true by definition of Ω_1. Furthermore, $\Omega_1^* \subset \Omega_1$ holds trivially, by definition of the sets constituting Ω_1^*.

It remains to prove $\Omega_1^* \supset \Omega_1$. Note that $\zeta > 0$ on Ω_1^*. Thus, $\lambda_F^A + \lambda_F^B = \bar{\lambda}$ on Ω_1^*. The remaining constraints are the shorting constraints of insiders ($\kappa \in [0,1]$), distressed investors ($\lambda_D^A \geq 0, \lambda_D^B \geq 0$), and financiers ($\lambda_F^A \geq 0, \lambda_F^B \geq 0$), a total of 6 constraints. To help characterize these constraints, note the following:

$$\{\tilde{f}_A(\kappa,\zeta) > 0,\, \tilde{d}_A(\kappa,\zeta) > 0\} = \{\lambda_F^A > 0,\, \lambda_D^A > 0\} \tag{85}$$
$$\{\tilde{f}_A(\kappa,\zeta) > 0,\, \tilde{d}_A(\kappa,\zeta) \leq 0\} = \{\lambda_F^A > 0,\, \lambda_D^A = 0\} \tag{86}$$
$$\{\tilde{f}_A(\kappa,\zeta) \leq 0,\, \tilde{d}_A(\kappa,\zeta) > 0\} = \{\lambda_F^A = 0,\, \lambda_D^A > 0\} \tag{87}$$
$$\{\tilde{f}_A(\kappa,\zeta) \leq 0,\, \tilde{d}_A(\kappa,\zeta) \leq 0\} = \{\lambda_F^A = 0,\, \lambda_D^A = 0\}, \tag{88}$$

[41]Correspondences are needed because of the possibility of multiple equilibria, which I have not ruled out in my proof. Multiple equilibria are captured mathematically by non-empty intersections of the sets defined above. Numerically, I have found parameterizations of the model in which $\Omega_{\zeta,F\not\to A} \cap \Omega_{\zeta}^{\kappa=0}$ and $\Omega_{\zeta,F\not\to B} \cap \Omega_{\zeta}^{\kappa=1}$ are non-empty. In those cases, I choose assign to (κ,ζ) the values dictated by $\Omega_{\zeta}^{\kappa=0}$ and $\Omega_{\zeta}^{\kappa=1}$.

and, for sector B,

$$\{\tilde{f}_B(\kappa,\zeta) > 0,\, \tilde{d}_B(\kappa,\zeta) > 0\} = \{\lambda_F^B > 0,\, \lambda_D^B > 0\} \tag{89}$$
$$\{\tilde{f}_B(\kappa,\zeta) > 0,\, \tilde{d}_B(\kappa,\zeta) \le 0\} = \{\lambda_F^B > 0,\, \lambda_D^B = 0\} \tag{90}$$
$$\{\tilde{f}_B(\kappa,\zeta) \le 0,\, \tilde{d}_B(\kappa,\zeta) > 0\} = \{\lambda_F^B = 0,\, \lambda_D^B > 0\} \tag{91}$$
$$\{\tilde{f}_B(\kappa,\zeta) \le 0,\, \tilde{d}_B(\kappa,\zeta) \le 0\} = \{\lambda_F^B = 0,\, \lambda_D^B = 0\}, \tag{92}$$

Importantly, $\{\lambda_F^A = 0, \lambda_D^A = 0\} = \{\kappa = 0\}$ and $\{\lambda_F^B = 0, \lambda_D^B = 0\} = \{\kappa = 1\}$, by funding market clearing. In addition, $\{\zeta > 0, \lambda_F^A = 0, \lambda_D^B = 0\} = \{\zeta > 0\, \lambda_F^B = 0, \lambda_D^A = 0\} = \varnothing$, by combining equations (66), (18), and (20). Consequently, the sets constituting Ω_1^* are exactly the intersection of $\{\zeta > 0\}$ with the pairwise combinations of the sets in (85)-(88) with the sets in (89)-(92), which are a completely exhaustive set of combinations, i.e., $\{\zeta > 0\} = \Omega_1^*$. Finally note, by the definition of ζ in the statement of Proposition C.4, that $\Omega \backslash \Omega_1 \subset \{\zeta = 0\}$ so that $\Omega_1 \subset \{\zeta > 0\}$. $\square$

C.4 Necessity of Leverage Constraints

To see the crucial role the leverage constraint plays in the results of Section 3, now suppose $\bar{\lambda} = +\infty$. Figure 14 shows financial distress is almost completely absent. Distressed investors rarely enter the market, lending spreads respond much more smoothly to changes in the state variables, and sector B spreads are minuscule across the state space.

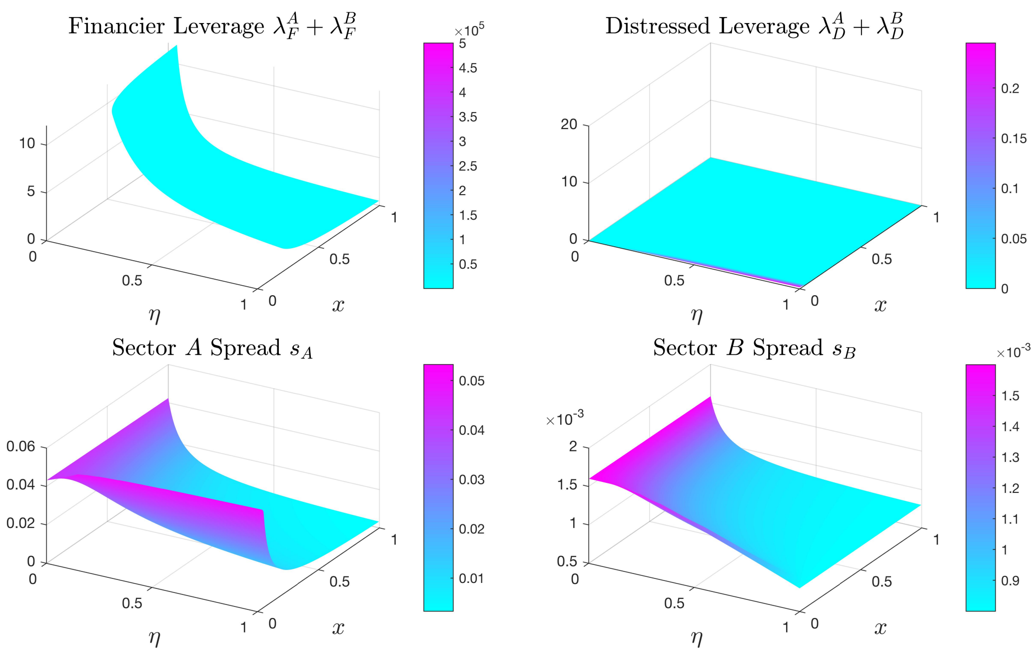

Figure 14: Equilibrium functions of (η, x) with $\alpha = 0.5$ fixed. Parameters: $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $G_A = G_B = 0.1$, $\Delta_A = 0.5$, $\Delta_B = 1$, $\rho = 0.02$, $\rho_F = 0.06$, $\chi = 0.05$, and $\bar{\lambda} = +\infty$.

The following proposition formalizes this result by characterizing when financial distress occurs.

Proposition C.5 (Distress without Leverage Constraints). *Consider Proposition C.3 with $\bar{\lambda} = +\infty$. Distressed investors lend to sector $z \in \{A, B\}$ if and only if financiers' wealth share $x_t\eta_t < \omega_{z,t}^*$, where*

$$\omega_{A,t}^* := \chi^{-1}\kappa_t\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2 \tag{93}$$
$$\omega_{B,t}^* := \chi^{-1}(1-\kappa_t)\phi_B(1-\Delta_B)^2\hat{\sigma}_B^2. \tag{94}$$

Proof of Proposition C.5. Let $\bar{\lambda} = +\infty$ so that $\zeta = 0$ in Proposition C.3. Specializing (66) to this case, we have Then, substituting s_A and π into λ_D^A in (20), we find $\lambda_D^A > 0$ if and only if $(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} > \chi$. Substituting $\hat{\pi}_{F\to A} := \kappa\phi_A(1-\Delta_A)\hat{\sigma}_A/(x\eta)$, this implies $\omega_A^* := \chi^{-1}\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2 > x\eta$. An identical analysis holds for sector B. □

Proposition C.5 illustrates the theoretical possibility of financial distress. If financiers' wealth is low relative to the amount of idiosyncratic risk they must bear, distressed investors have an incentive to enter the market. These incentives are summarized by the thresholds (ω_A^*, ω_B^*). That said, even for moderate diversification, these thresholds are tiny. Consider the case of symmetric sectors, such that $\kappa_t = 0.5$. Under $\chi = 0.05$ and $\hat{\sigma}_A = \hat{\sigma}_B = 0.2$, $\phi_A = \phi_B = 0.5$, and $\Delta_A = \Delta_B = 0.5$, we have $\omega_{A,t}^* = \omega_{B,t}^* = 0.05$. If financiers hold more than 5% of total wealth, distress is impossible.

Furthermore, as $\Delta_A, \Delta_B \to 1$, distressed investors never take positive positions, as (93)-(94) show. Under perfect diversification, financiers can perfectly hedge all the risks on their funding portfolio, so their leverage decisions are completely decoupled from the risks they must bear. Less efficient lenders never enter if they can finance more efficient lenders to do the same. This result explains why models without leverage constraints, such as Brunnermeier and Sannikov (2014), feature inefficiency that falls with financiers' fundamental risks.

Proof of Proposition 3.1. Suppose $\lambda_F^A + \lambda_F^B = \bar{\lambda}$, but $\lambda_D^A + \lambda_D^B = 0$. The latter, plus funding market clearing, implies $\lambda_F^A = \kappa\phi_A/x\eta$ and $\lambda_F^B = (1-\kappa)\phi_B/x\eta$. Summing these results yields $\kappa\phi_A + (1-\kappa)\phi_B = \bar{\lambda}x\eta$. If $\phi_A = \phi_B$, then this implies $x\eta = \bar{\lambda}^{-1}$, which contradicts $d(x_t\eta_t) \neq 0$ (if $\sigma_A = \sigma_B \equiv 0$, then time-dynamics are non-zero outside of steady-state; otherwise, $x\eta$ is diffusive). This proves part (i).

Now, suppose $\lambda_F^A + \lambda_F^B < \bar{\lambda}$ so that $\zeta = 0$. Then, using the stated assumption of part (ii) and equation (18), we have $\chi \geq \bar{\lambda}(1-\Delta_A)^2\hat{\sigma}_A^2 \geq \lambda_F^A(1-\Delta_A)^2\hat{\sigma}_A^2 \geq s_A - \sigma_A \cdot \pi$, so that $\lambda_D^A = 0$ by (20). Repeating this analysis for sector B proves part (ii). □

C.5 Endogenous Busts and Financial Instability

In the following proof of Proposition 3.2, we assume the following condition:

$$(1-\phi/\bar{\lambda})^2\Big[\rho - \rho_F + \bar{\lambda}\chi\Big] + 2\delta(1-\phi/\bar{\lambda})\Big[(\nu_F\bar{\lambda}/\phi)^{1/2} - 1\Big] - (1-\phi)^2\hat{\sigma}^2 < 0 < \chi. \tag{95}$$

Proof of Proposition 3.2. First, note that the stated assumptions imply $\kappa = \alpha = 1/2$ at all times. Next, $\Delta = \Delta_\tau$ large enough implies the leverage constraint (17) binds in the future. Indeed, Δ large enough guarantees

$$\chi > \frac{1}{2}\bar{\lambda}(1-\Delta)^2\hat{\sigma}^2 \tag{96}$$

holds. Using part (i) of Proposition 3.1, inequality (96) implies that $\lambda_D^A + \lambda_D^B > 0$ if and only if $\lambda_F^A + \lambda_F^B = \bar{\lambda}$. Hence, using equations (79) and (81) to compute the drift of $\log(x\eta)$ away from the constraint, and substituting $\lambda_D^A = \lambda_D^B = 0$, we have

$$\mu^{\log(x\eta)} = \underbrace{(1-x\eta)(\rho-\rho_F) + \frac{1}{2}(1-x\eta)(\frac{\phi(1-\Delta)\hat{\sigma}}{x\eta})^2 - \frac{(1-\phi)^2\hat{\sigma}^2}{1-\eta}}_{:=\mu_1} + \underbrace{\frac{\delta}{x\eta}[\eta\frac{\nu_F}{\nu_F+\nu_D} + x(\nu_F+\nu_D) - 2x\eta]}_{:=\mu_2}.$$

By maximizing μ_1 and μ_2 (separately) over (x,η), subject to $x\eta > \phi/\bar{\lambda}$, it is straightforward to show that the corner solution $x\eta = \phi/\bar{\lambda}$ is maximal for each. The result is

$$\sup_{x,\eta\,:\,x\eta>\phi/\bar{\lambda}} \mu^{\log(x\eta)} < (1-\phi/\bar{\lambda})\Big[\rho-\rho_F + \frac{1}{2}(\bar{\lambda}(1-\Delta)\hat{\sigma})^2\Big] + 2\delta\Big[(\nu_F\bar{\lambda}/\phi)^{1/2} - 1\Big] - \frac{(1-\phi)^2\hat{\sigma}^2}{1-\phi/\bar{\lambda}} < 0,$$

where the second inequality holds by (95)-(96). This shows that if Δ_τ is made large enough, then $\log(x_{\tau+t}\eta_{\tau+t})$ must have a negative drift at least until the leverage constraint (17) is hit, i.e., for a length of time $T := \inf\{t : x_{\tau+t}\eta_{\tau+t} = \phi/\bar{\lambda}\}$.

Immediately after the constraint is hit, the drift is still negative. Indeed, using Proposition C.4, we may compute $\zeta_{\tau+T} = \chi - \frac{1}{2}\bar{\lambda}(1-\Delta)^2\hat{\sigma}^2$ (take $\tilde{\zeta}^*$ from the proposition, then substitute $\phi_A = \phi_B = \phi$ and $x\eta = \phi/\bar{\lambda}$), which is positive by (96). Substituting into spreads from Proposition C.3, we obtain $s_{\tau+T} = \chi$ in both sectors. We may then calculate

$$\begin{aligned}\mu_{\tau+T}^{\log(x\eta)} &= (1-x\eta)[\rho-\rho_F+\bar{\lambda}\chi] - \frac{(1-\phi)^2\hat{\sigma}^2}{1-\eta} + \frac{\delta}{x\eta}[\eta\frac{\nu_F}{\nu_F+\nu_D} + x(\nu_F+\nu_D) - 2x\eta] \\ &< (1-x\eta)[\rho-\rho_F+\bar{\lambda}\chi] - \frac{(1-\phi)^2\hat{\sigma}^2}{1-x\eta} + \frac{\delta}{x\eta}[\eta\frac{\nu_F}{\nu_F+\nu_D} + x(\nu_F+\nu_D) - 2x\eta]\end{aligned}$$

Substituting $x\eta = \phi/\bar{\lambda}$, and recalling that $\max_{x\eta=\phi/\bar{\lambda}} \mu_1 = 2\delta[(\nu_F\bar{\lambda}/\phi)^{1/2} - 1]$, we obtain

$$\mu_{\tau+T}^{\log(x\eta)} < (1-\phi/\bar{\lambda})[\rho-\rho_F+\bar{\lambda}\chi] - \frac{(1-\phi)^2\hat{\sigma}^2}{1-\phi/\bar{\lambda}} + 2\delta\Big[(\nu_F\bar{\lambda}/\phi)^{1/2} - 1\Big] < 0,$$

with the latter inequality again by (95).

Finally, equity market clearing says $\eta(1-x)\lambda_D^A = \eta(1-x)\lambda_D^B = \frac{1}{2}\phi - \eta x\frac{1}{2}\bar{\lambda}$. Since spreads are equal to χ, we have $\lambda_{D,\tau+T}^A = \lambda_{D,\tau+T}^B = 0$ but $\lambda_{D,\tau+T+\varepsilon}^A > 0$ and $\lambda_{D,\tau+T+\varepsilon}^B > 0$ for $\varepsilon > 0$ small enough. Consequently, distress costs

$$\chi\eta_{\tau+T+\varepsilon}(1-x_{\tau+T+\varepsilon})(\lambda_{D,\tau+T+\varepsilon}^A + \lambda_{D,\tau+T+\varepsilon}^B) > 0.$$

This proves that an endogenous bust (Definition 2) occurs. □

Proof of Proposition 3.3. First, note that $Y = G - \chi\eta(1-x)(\lambda_D^A + \lambda_D^B)$ so that

$$\mathrm{Var}_t[dY_t] = \chi^2\Big(\eta_t^2(1-x_t)^2\mathrm{Var}_t[d(\lambda_{D,t}^A + \lambda_{D,t}^B)] + (\lambda_{D,t}^A + \lambda_{D,t}^B)^2\mathrm{Var}_t[d(\eta_t(1-x_t))]\Big).$$

With Δ_A, Δ_B chosen high enough, the conditions of Proposition 3.1 hold so that $\lambda_D^A + \lambda_D^B > 0$ if and only if $\lambda_F^A + \lambda_F^B = \bar{\lambda}$. Therefore, $\mathrm{Var}_t[dY_t] = 0$ when the leverage constraint is slack and $\mathrm{Var}_t[dY_t] > 0$ otherwise. □

D Other Financial Shocks

This section presents model extensions required to analyze other financial shocks. Each shock has a theoretical proposition associated to it, the proofs of which are contained at the end of this section (see Appendix D.6).

D.1 LTV Shock

Another important financial shock is an increase in ϕ, which reduces the idiosyncratic risk insiders must bear when investing in capital. Like a loan-to-value ratio, ϕ is the fraction of assets that insiders can borrow against, so I refer to this shock as an *LTV shock*. This type of shock is widely studied in the quantitative modeling literature, with somewhat disparate results.[42] Here, I study the implications of this shock in my model. We have the following.

Proposition D.1. *Consider the equilibrium of Proposition 2.3. Suppose at time t the economy is sufficiently close to an interior steady state and then there is a small increase in ϕ_A. Sector-A capital share increases, $\kappa_t > \kappa_{t-}$, if and only if sector-A spreads are less than the total sector-A risk premium (i.e., $s_A < G_A - r$). If either (i) sectors are nearly symmetric or (ii) $|\log(\kappa_t/\kappa_{t-})|$ is not too large, then $\mu_t^\eta > \mu_{t-}^\eta$.*

The key point to note about ϕ is that it is a risk transfer between insiders and financiers. Because financiers are better diversified than insiders, this risk transfer is value-enhancing and generates sectoral reallocation. Mathematically, equation (15) shows that higher ϕ_A lowers sector A's idiosyncratic risk premia, which are equal to

$$\text{idio rp}_A = \kappa\Big[\frac{(1-\phi_A)^2}{\alpha(1-\eta)} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta}\Big]\hat{\sigma}_A^2.$$

This quantity is decreasing in ϕ_A for a well-diversified sector.

That said, the risk transfer to financiers shifts idiosyncratic risk compensation from insiders to financiers. In response to the LTV shock, lending spreads increase, which is why LTV shocks are sometimes interpreted as "credit demand shocks." Thus, an increase in ϕ_A unambiguously raises financier profitability. Although short-run financier leverage $\eta^{-1}[\kappa\phi_A + (1-\kappa)\phi_B]$ can increase with ϕ_A, the effect on long-run financier leverage is ambiguous through the slow rise in η.

D.2 Capital-Requirement Shock

Another possible finance-centric explanation for boom-bust cycles is improved financier access to outside equity. Perhaps financiers are equity-issuance constrained, perhaps because of capital requirements or more fundamental agency frictions. A relaxation in capital requirements improves financiers' risk-sharing with the rest of the economy. To model this scenario, I allow financiers to partially issue equity against their assets, requiring them to keep $1-\phi_F$ fraction of skin in the game, like the insiders of sectors A and B.[43] Shocks to the parameter ϕ_F can be called *capital requirement shocks*. We have the following result.

Proposition D.2. *Consider the equilibrium of Proposition 2.3, with capital requirement $1-\phi_F$. If $\Delta := \Delta_A \equiv \Delta_B$, then ϕ_F-shocks and Δ-shocks are equivalent in the following sense: the equilibrium only depends on $\Delta^* := 1-(1-\Delta)(1-\phi_F)$ and not ϕ_F or Δ independently.*

Capital-requirement shocks (ϕ_F) are similar to diversification shocks (Δ) in that both provide ways for financiers to diversify idiosyncratic risks. For this reason, both parameters appear together in the expression for financiers' idiosyncratic risk prices, i.e.,

$$\hat{\pi}_{F\to A} = \frac{\kappa\phi_A(1-\Delta_A)(1-\phi_F)\hat{\sigma}_A}{\eta} \quad \text{and} \quad \hat{\pi}_{F\to B} = \frac{(1-\kappa)\phi_B(1-\Delta_B)(1-\phi_F)\hat{\sigma}_B}{\eta}.$$

[42] See, for example, Kiyotaki et al. (2011), Justiniano et al. (2015b), Favilukis et al. (2017), and Kaplan et al. (2017).

[43] This outside equity is assumed to be pooled, thus perfectly diversified, and sold to the market. The equilibrium of this modified economy is detailed in the appendix.

Indeed, Proposition D.2 shows that looser capital requirements act like broad, sectorally-agnostic increases in diversification. It follows that looser capital requirements will generate financial leverage. But a key distinction is that ϕ_F applies symmetrically to both sectors, whereas Δ_A, Δ_B can be asymmetric. Although a sector-specific diversification shock generates a reallocation, looser capital requirements will tend to raise asset prices and allocations across the board, as would a broad diversification improvement.

This is empirically relevant. Referring back to the motivational figure 1, we see household credit rose as a *share* of total private non-financial credit. From the multi-asset perspective, diversification shocks are more likely to generate these features than a general capital-requirement shock. Prior papers, such as Justiniano et al. (2015a), adopt reduced-form credit-supply shocks that relax "lending constraints," as a plausible explanation for why house prices rose. But in those papers, the only positive net supply asset is housing, so they cannot explain why house prices might have risen more than other assets.

D.3 Risk-Tolerance Shock

A popular culprit of boom-bust cycles has been excessive optimism or excessive risk tolerance, e.g., Kindleberger (1978). Because of the nature of asset pricing, beliefs and risk tolerance always enter risk premia jointly. I thus consider shocks to risk tolerance in this section.

Now, agents are endowed with recursive utility as in equation (46) in the appendix. See Appendix B.5 for details on solving agents' portfolio problems under these preferences. For simplicity, I assume all agents have unitary elasticity of intertemporal substitution, but they differ in their risk-aversion parameters, $\gamma_A, \gamma_B, \gamma_F$. Risk-tolerance shocks are shocks to these parameters individually.

Proposition D.3. *Consider equilibrium with risk aversions γ_A, γ_B, and γ_F. Assume sectors A and B are symmetric, including $\gamma_A = \gamma_B \equiv \gamma$. Suppose at time t the economy is at steady state and then one of the following occurs:*

(i) γ_A decreases by a small amount. Then, $\kappa_t > \kappa_{t-}$ and $\mu_t^\eta > \mu_{t-}^\eta$.

(ii) γ_F decreases by a small amount. Then, $\kappa_t = \kappa_{t-}$ and $\mu_t^\eta < \mu_{t-}^\eta$.

(iii) γ_A and γ_F both decrease by the same small amount. Then, $\kappa_t > \kappa_{t-}$ and $\mu_t^\eta < \mu_{t-}^\eta$, the latter if and only if $\gamma > \gamma_F \alpha_{t-}$.

Intuitively, a decrease in γ_A lowers discount rates in sector A, which generates a sectoral allocation. However, with lower discount rates, insiders are willing to pay higher spreads to financiers, increasing their long-run wealth share. In this sense, a γ_A-shock is a credit-demand shock, just like the LTV shock to ϕ_A. A decrease in γ_F is a credit-supply shock, because it lowers required lending spreads. But because γ_F applies symmetrically to both sectors, lending spreads decrease across the board. A sectoral reallocation is less likely, as with the capital requirement shock to ϕ_F. Only if γ_A and γ_F both decrease, with γ_B left unchanged, can the model generate both reallocation and leverage.[44]

D.4 Uncertainty Shock

Uncertainty shocks have been proposed as a possible driver of cycles: when uncertainty is low, banks may take greater leverage, and the economy suffers when uncertainty reverts. A sectoral uncertainty shock would be a reduction in $\hat{\sigma}_A$. We have the following result, which shows that lower sectoral uncertainty generates a reallocation but may not generate financier leveraging.

Proposition D.4. *Consider the equilibrium of Proposition 2.3. Suppose at time t the economy is in steady state and then $\hat{\sigma}_A$ decreases by a small amount. Then, $\kappa_t > \kappa_{t-}$. If either (i) sectors are symmetric or (ii) $\hat{\sigma}_A > \hat{\sigma}_B = 0$, then $\mu_t^\eta = \mu_{t-}^\eta$.*

[44] If sector A is interpreted as housing, such a shock corresponds most closely to the survey evidence in Case and Shiller (2003) and the evidence in Foote et al. (2012). Kaplan et al. (2017) and Glaeser and Nathanson (2017) have model economies where agents become optimistic only about housing. Even though Landvoigt (2016) incorporates securitization, a key element of his story is the underpricing of mortgage risk by lenders.

To understand this result, consider a hypothetical economy with no diversification ($\Delta_A = \Delta_B = 0$) but two values of idiosyncratic volatility that apply to insiders and financiers separately, i.e., $\hat{\sigma}_{A,A}$ and $\hat{\sigma}_{A,F}$. The economy is otherwise exactly identical. One can show the equilibrium of this economy is isomorphic to the equilibrium of Proposition 2.3, if $\hat{\sigma}_{A,A} = \hat{\sigma}_A$ and $\hat{\sigma}_{F,A} = (1-\Delta_A)\hat{\sigma}_A$. Therefore, a diversification shock operates by lowering $\hat{\sigma}_{F,A}$ and keeping $\hat{\sigma}_{A,A}$ fixed.

An uncertainty shock has the effect of lowering both $\hat{\sigma}_{F,A}$ and $\hat{\sigma}_{A,A}$ proportionally. The result of this type of shock is to scale down all agents' idiosyncratic risk premia equally. The long-run effect of low idiosyncratic uncertainty is ambiguous in the sense that η_t could be higher or lower, precisely because both insiders and financiers are affected.[45]

D.5 Foreign-Savings Shock

A final alternative to consider is an increase in demand for safe assets, which tends to reduce interest rates and may fuel the boom, e.g., Bernanke (2005). Because much of this safe-asset demand manifested empirically as foreign agents buying US Treasury securities and other close substitutes, I call this a *foreign-savings shock*. This is also consistent with the documented increase in foreign demand for highly-rated securitized products, which behave like safe assets.

To model foreign savings, I introduce a wedge into the bond-market-clearing condition, which now becomes

$$N_{A,t} + N_{B,t} + N_{F,t} + N_t^* = K_t.$$

I assume N_t^* follows some exogenous deterministic process, which is co-integrated with capital K_t. A foreign-savings shock can be modeled as an exogenous change to N_t^*. Note that foreign savings also affects the goods market, because net interest payments to foreigners must come out of consumption. This modified economy has three state variables, the relative wealth between financiers, insiders, and foreigners:

$$\alpha_t := \frac{N_{A,t}}{N_{A,t}+N_{B,t}}, \quad \eta_t := \frac{N_{F,t}}{N_{F,t}+N_{A,t}+N_{B,t}}, \quad \text{and} \quad \eta_t^* := \frac{N_t^*}{K_t}.$$

We have the following result.

Proposition D.5. *Consider equilibrium with foreign savings. Assume $G_A = G_B$. Suppose at time t the economy is at steady state and then $\eta_t^* - \eta_{t-}^* > 0$. Then, $\kappa_t = \kappa_{t-}$ and $\eta_t = \eta_{t-}$, but financier leverage grows by $(1-\eta_{t-}^*)/(1-\eta_t^*) > 1$.*

The key to Proposition D.5 is that foreign inflows raise all domestic agents' leverage proportionally. Foreign savings of η_t^* per unit of domestic wealth result in leverage of $(1-\eta_t^*)^{-1}$ for the domestic representative agent. In particular, financier leveraging does occur after a foreign-savings shock.

But leverage is distributed equally across all domestic agents. As a result, all idiosyncratic risk prices are given by the formulas in (12)-(13), with an additional scaling by $(1-\eta_t^*)^{-1}$. Formulas (10)-(11) then show the dynamics of (α, η) are merely scaled by $(1-\eta_t^*)^{-2}$, explaining why η_t is unaffected by foreign savings near the steady state. Applying this logic to formula (14) also explains why κ_t is unaffected by foreign savings. Intuitively, foreign savings are not directed toward any particular sector, so reallocation does not occur.

D.6 Proofs

For all proofs below, marginal calculations are sufficient given the shocks are small enough. In addition, Lemma 2.5 allows us to ignore any effects on the levels of state variables (α, η). For statements about the effect on μ^η within the equilibrium of Proposition 2.3, note that for any parameter $\mathbf{p}$, we have

$$\frac{d\mu^\eta}{d\mathbf{p}} = \eta(1-\eta)\Big(\frac{\partial\big[\hat{\pi}_{F\to A}^2 + \hat{\pi}_{F\to B}^2 - \alpha\hat{\pi}_A^2 - (1-\alpha)\hat{\pi}_B^2\big]}{\partial\mathbf{p}} + \frac{2}{1-\kappa}\Big[\hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_A^2 - \frac{\kappa\mu^\eta}{\eta(1-\eta)}\Big]\frac{d\log\kappa}{d\mathbf{p}}\Big). \tag{97}$$

[45] This speaks to an important difference between how I am modeling the financial sector and how it has been modeled in the literature. Because both financiers and insiders are taking idiosyncratic risks, they both demand idiosyncratic risk compensation that rises with higher uncertainty. By contrast, Di Tella (2017) assumes uncertainty shocks only affect the balance sheets of financiers.

Proof of Proposition D.1. From equation (14), compute $d\kappa/d\phi_A \propto \hat{\pi}_A - (1-\Delta_A)\hat{\pi}_{F\to A}$. The right-hand-side is proportional to $G_A - r - s_A$, since by (15) we have $G_A - r - s_A = \phi_A\hat{\sigma}_A\hat{\pi}_A - \phi_A s_A = \phi_A\hat{\sigma}_A[\hat{\pi}_A - (1-\Delta_A)\hat{\pi}_{F\to A}]$.

For μ^η, use equation (97) with $\mathbf{p} = \phi_A$ to get

$$\frac{d\mu^\eta}{d\phi_A} = 2\eta(1-\eta)\Big[\underbrace{\Big(\frac{(1-\Delta_A)\hat{\pi}_{F\to A}}{\eta} + \frac{\hat{\pi}_A}{1-\eta}\Big)\kappa\hat{\sigma}_A}_{:=D_1} + \underbrace{\frac{1}{1-\kappa}\Big(\hat{\pi}^2_{F\to A} - \alpha\hat{\pi}^2_A - \frac{\kappa\mu^\eta}{\eta(1-\eta)}\Big)\frac{d\log\kappa}{d\phi_A}}_{:=D_2}\Big]$$

Note that $D_1 > 0$. Under either of assumptions (i) or (ii), $D_2 \approx 0$. Indeed, given the assumption that the economy is near enough to steady-state, we can ignore the μ^η term and additionally establish $\hat{\pi}^2_{F\to A} - \alpha\hat{\pi}^2_A \approx (1-\alpha)\hat{\pi}^2_{F\to A} - \alpha\hat{\pi}^2_{F\to B}$. Under near-symmetry (i) this term is approximately zero. Under $|\log(\kappa_t/\kappa_{t-})|$ not too large (ii), $d\log\kappa/d\phi_A$ must be nearly zero. Hence, $d\mu^\eta/d\phi_A > 0$ under (i) or (ii). □

Proof of Proposition D.2. Repeat the steps of Proposition 2.3 with λ^z replaced by $(1-\phi_F)\lambda^z$, with funding market clearing replaced by $(1-\phi_F)\int_{i-\Delta_z}^{i}\Delta_z^{-1}[\lambda^z_{j,t}n^F_{j,t}dj = \phi_z k^z_{i,t}$, and with aggregate risk market clearing replaced by $\int_0^1[\theta^A_{i,t}n^A_{i,t} + \theta^B_{i,t}n^B_{i,t} + \theta^F_{i,t}n^F_{i,t}]di = (1-\phi_F)\int_0^1(\lambda^A_{i,t}\sigma_A + \lambda^B_{i,t}\sigma_B)n^F_{i,t}di.$ □

Proof of Proposition D.3. Using the results of Appendix B.5, we can derive the equilibrium with risk aversions $\gamma_A, \gamma_B, \gamma_F$ and unitary EIS. The following is a sketch of the proof, following the same line of argument as Proposition C.3. First, all agents consume ρ fraction of their wealth. Financiers' optimal portfolios are $\lambda_z = s_A/\gamma_F(1-\Delta_z)^2\hat{\sigma}_z^2$ for $z \in \{A, B\}$, whereas insiders' optimal portfolios lead to conditions $\kappa/\alpha(1-\eta) \geq [G_A - \phi_A s_A - r]/\gamma_A(1-\phi_A)^2\hat{\sigma}_A^2$ and $(1-\kappa)/(1-\alpha)(1-\eta) \geq [G_B - \phi_B s_B - r]/\gamma_B(1-\phi_B)^2\hat{\sigma}_B^2$, respectively. Define the following modified idiosyncratic risk prices,

$$\hat{\pi}_{F\to A} = \gamma_F\frac{\kappa\phi_A(1-\Delta_A)\hat{\sigma}_A}{\eta} \quad \text{and} \quad \hat{\pi}_{F\to B} = \gamma_F\frac{(1-\kappa)\phi_B(1-\Delta_B)\hat{\sigma}_B}{\eta}$$
$$\hat{\pi}_A = \gamma_A\frac{\kappa(1-\phi_A)\hat{\sigma}_A}{\alpha(1-\eta)} \quad \text{and} \quad \hat{\pi}_B = \gamma_B\frac{(1-\kappa)(1-\phi_B)\hat{\sigma}_B}{(1-\alpha)(1-\eta)}.$$

In terms of these risk prices, optimal portfolios can be written $\lambda_z = \hat{\pi}_{F\to z}/\gamma_F(1-\Delta_z)\hat{\sigma}_z$, $\kappa/\alpha(1-\eta) = \hat{\pi}_A/\gamma_A(1-\phi_A)\hat{\sigma}_A$, and $(1-\kappa)/(1-\alpha)(1-\eta) = \hat{\pi}_B/\gamma_B(1-\phi_B)\hat{\sigma}_B$; equilibrium expected excess returns can be written $s_z = (1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}$ and $G_z - \phi_z s_z - r = (1-\phi_z)\hat{\sigma}_z\hat{\pi}_z$ for $z \in \{A, B\}$. Consequently, wealth share dynamics are given by

$$\mu^\alpha = \alpha(1-\alpha)\Big[\frac{\hat{\pi}_A^2}{\gamma_A} - \frac{\hat{\pi}_B^2}{\gamma_B}\Big] \tag{98}$$
$$\mu^\eta = \eta(1-\eta)\Big[\frac{\hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}}{\gamma_F} - \alpha\frac{\hat{\pi}_A^2}{\gamma_A} - (1-\alpha)\frac{\hat{\pi}_B^2}{\gamma_B}\Big]. \tag{99}$$

Finally, the equilibrium capital share is derived by taking the difference between insiders' FOCs in sectors A and B:

$$\kappa = \min(1, \max(0, \tilde{\kappa})),$$
$$\text{where} \quad \tilde{\kappa} := \frac{G_A - G_B + \big[\gamma_B\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \gamma_F\frac{\phi_B^2(1-\Delta_B)^2}{\eta}\big]\hat{\sigma}_B^2}{\big[\gamma_A\frac{(1-\phi_A)^2}{\alpha(1-\eta)} + \gamma_F\frac{\phi_A^2(1-\Delta_A)^2}{\eta}\big]\hat{\sigma}_A^2 + \big[\gamma_B\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \gamma_F\frac{\phi_B^2(1-\Delta_B)^2}{\eta}\big]\hat{\sigma}_B^2}. \tag{100}$$

Given this equilibrium, we now perform comparative statics. For any derivatives stated with respect to γ^{-1} rather than $\gamma \in \{\gamma_A, \gamma_B, \gamma_F\}$, simply flip the signs derived below.

First, differentiate $\tilde{\kappa}$ with respect to each of the risk aversions, which using the fact that κ is assumed interior, leads to the following (all with the same constant of proportionality):

$$\frac{d\kappa}{d\gamma_A} \propto -(1-\phi_A)\hat{\sigma}_A \gamma_A^{-1} \hat{\pi}_A < 0$$
$$\frac{d\kappa}{d\gamma_B} \propto (1-\phi_B)\hat{\sigma}_B \gamma_B^{-1} \hat{\pi}_B > 0$$
$$\frac{d\kappa}{d\gamma_F} \propto \phi_B(1-\Delta_B)\hat{\sigma}_B \gamma_F^{-1} \hat{\pi}_{F\to B} - \phi_A(1-\Delta_A)\hat{\sigma}_A \gamma_F^{-1} \hat{\pi}_{F\to A}.$$

Under the assumption that sectors are symmetric, the last expression is exactly zero.

The modification of (97) for this environment is (where $\gamma \in \{\gamma_F, \gamma_A, \gamma_B\}$ is one of the risk aversions in question):

$$\frac{d\mu^\eta}{d\gamma} = \eta(1-\eta)\Big(\frac{\partial\big[\frac{\hat{\pi}_{F\to A}^2 + \hat{\pi}_{F\to B}^2}{\gamma_F} - \alpha\frac{\hat{\pi}_A^2}{\gamma_A} - (1-\alpha)\frac{\hat{\pi}_B^2}{\gamma_B}\big]}{\partial\gamma} + D\frac{d\log\kappa}{d\gamma}\Big),$$
$$\text{where} \quad D := \frac{2}{1-\kappa}\Big[\frac{\hat{\pi}_{F\to A}^2}{\gamma_F} - \alpha\frac{\hat{\pi}_A^2}{\gamma_A} - \frac{\kappa\mu^\eta}{\eta(1-\eta)}\Big].$$

The results are

$$\frac{d\mu^\eta}{d\gamma_A} = \eta(1-\eta)\Big(-\alpha(\frac{\hat{\pi}_A}{\gamma_A})^2 + D\frac{d\log\kappa}{d\gamma_A}\Big)$$
$$\frac{d\mu^\eta}{d\gamma_B} = \eta(1-\eta)\Big(-(1-\alpha)(\frac{\hat{\pi}_B}{\gamma_B})^2 + D\frac{d\log\kappa}{d\gamma_B}\Big)$$
$$\frac{d\mu^\eta}{d\gamma_F} = \eta(1-\eta)\Big((\frac{\hat{\pi}_{F\to A}}{\gamma_F})^2 + (\frac{\hat{\pi}_{F\to B}}{\gamma_F})^2 + D\frac{d\log\kappa}{d\gamma_F}\Big).$$

Under the assumption that the economy begins in steady state, $\mu^\eta = 0$ so that $D = \frac{2}{1-\kappa}\gamma_F^{-1}\big[(1-\alpha)\hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_{F\to B}^2\big]$. Under the assumption that sectors are symmetric $\hat{\pi}_{F\to A} = \hat{\pi}_{F\to B}$ so that $D = 0$. This proves $d\mu^\eta/d\gamma_A < 0$, $d\mu^\eta/d\gamma_B < 0$, and $d\mu^\eta/d\gamma_F > 0$.

For part (iii), sum the results across insiders and financiers, using the symmetry condition to obtain $d\kappa/d\gamma_A + d\kappa/d\gamma_F < 0$ and as long as $\gamma_A > \alpha\gamma_F = \gamma_F/2$, we have $d\mu^\eta/d\gamma_A + d\mu^\eta/d\gamma_F > 0$. This completes the proof. □

Proof of Proposition D.4. First, from equation (14), compute $d\kappa/d\hat{\sigma}_A \propto -2\phi_A(1-\Delta_A)\hat{\pi}_{F\to A} < 0$.

For μ^η, use equation (97) with $\mathbf{p} = \hat{\sigma}_A$ to get

$$\frac{d\mu^\eta}{d\hat{\sigma}_A} = 2\eta(1-\eta)\Big[\Big(\hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_A^2\Big)\hat{\sigma}_A^{-1} + \frac{1}{1-\kappa}\Big(\hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_A^2 - \frac{\kappa\mu^\eta}{\eta(1-\eta)}\Big)\frac{d\log\kappa}{d\hat{\sigma}_A}\Big]$$

Given the assumption that the economy is initially at steady-state, $\mu^\eta = 0$. Hence, $d\mu^\eta/d\hat{\sigma}_A \propto \hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_A^2$. Under either of assumptions (i) or (ii), this quantity is zero. Indeed, given the assumption that the economy is at steady-state, $\hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_A^2 = (1-\alpha)\hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_{F\to B}^2$. Under symmetry (i) this term is exactly zero. Under $\hat{\sigma}_B = 0$ (ii), we have $\mu^\eta = \eta(1-\eta)[\hat{\pi}_{F\to A}^2 - \alpha\hat{\pi}_A^2] = 0$. □

Proof of Proposition D.5. Given the assumption that N_t^* is co-integrated with capital K_t, the economy begins with a steady-state value $\eta^* = N_t^*/K_t$. Repeating the same steps as in the proof of Proposition C.3, we find that all idiosyncratic risk prices simply require an additional $1-\eta^*$ in the denominator Thus, risk prices (12)-(13) are replaced by

$$\hat{\pi}_{F\to A} = \frac{\kappa\phi_A(1-\Delta_A)\hat{\sigma}_A}{(1-\eta^*)\eta} \quad \text{and} \quad \hat{\pi}_{F\to B} = \frac{(1-\kappa)\phi_B(1-\Delta_B)\hat{\sigma}_B}{(1-\eta^*)\eta}$$
$$\hat{\pi}_A = \frac{\kappa(1-\phi_A)\hat{\sigma}_A}{(1-\eta^*)\alpha(1-\eta)} \quad \text{and} \quad \hat{\pi}_B = \frac{(1-\kappa)(1-\phi_B)\hat{\sigma}_B}{(1-\eta^*)(1-\alpha)(1-\eta)}.$$

With this replacement, the expressions (10)-(11) for μ^α and μ^η continue to hold. Both of these drifts therefore scale with $(1-\eta^*)^{-2}$. As a result, when the economy is in steady state, there is identically zero sensitivity of μ^η to η^*. Furthermore, the equilibrium capital share is now given by

$$\kappa = \min(1, \max(0, \tilde{\kappa})),$$
$$\text{where} \quad \tilde{\kappa} := \frac{(G_A - G_B)(1-\eta^*) + \left[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta}\right]\hat{\sigma}_B^2}{\left[\frac{(1-\phi_A)^2}{\alpha(1-\eta)} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta}\right]\hat{\sigma}_A^2 + \left[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta}\right]\hat{\sigma}_B^2},$$

which has zero sensitivity to η^* when $G_A = G_B$. Finally, financier leverage is computed as $\lambda^A + \lambda^B = \frac{\kappa\phi_A + (1-\kappa)\phi_B}{\eta(1-\eta^*)}$. Combined with the previous results, this shows that $\frac{\lambda_t^A + \lambda_t^B}{\lambda_{t-}^A + \lambda_{t-}^B} = \frac{1-\eta_{t-}^*}{1-\eta_t^*}$. $\square$

E Extensions and Auxiliary Results

E.1 Differentiated Goods

For analytical tractability, I have assumed that the consumption goods of sectors A and B are perfect substitutes. In this appendix, I allow the goods to be differentiated as a robustness exercise. In particular, I replace agents' utility functions (1) with

$$\mathcal{U}_t := \mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(s-t)} \log(c_s)ds\Big],$$

where $c := a^{1-\beta}b^\beta$ is a Cobb-Douglas aggregate of the sector A good a and the sector B good b. Cobb-Douglas implies the two consumption goods have expenditure shares of $1-\beta$, β. I assume the composite good c is the numeraire. Let the relative prices of a and b be p_A and p_B. All other features of the model are unchanged.

With this modification, the equilibrium of Section 2 is modified as follows. First, the resource constraint from Definition 1 must replaced by three goods market clearing conditions:

$$\int_0^1 G_A k_{i,t}^A di = \int_0^1 [a_{i,t}^A + a_{i,t}^B + a_{i,t}^F]di$$
$$\int_0^1 G_B k_{i,t}^B di = \int_0^1 [b_{i,t}^A + b_{i,t}^B + b_{i,t}^F]di$$
$$\int_0^1 (G_A k_{i,t}^A)^{1-\beta}(G_B k_{i,t}^B)^\beta di = \int_0^1 [c_{i,t}^A + c_{i,t}^B + c_{i,t}^F]di + \int_0^1 [\iota_{i,t}^A k_{i,t}^A + \iota_{i,t}^B k_{i,t}^B]di.$$

The third condition aggregates output into the numeraire basket and splits this output into consumption and investment, which I assume is denominated in units of the numeraire.

Second, the equilibrium capital share κ and total capital growth ι are now determined via

$$\begin{aligned}\rho\Big[\frac{1-\beta}{\kappa} - \frac{\beta}{1-\kappa}\Big] &- (\kappa\|\sigma_A\|^2 - (1-\kappa)\|\sigma_B\|^2) \\ &= \Big[(1-\phi_A)\hat{\pi}_A + \phi_A(1-\Delta_A)\hat{\pi}_{F\to A}\Big]\hat{\sigma}_A - \Big[(1-\phi_B)\hat{\pi}_B + \phi_B(1-\Delta_B)\hat{\pi}_{F\to B}\Big]\hat{\sigma}_B\end{aligned} \tag{101}$$

and

$$\iota = (G_A\kappa)^{1-\beta}(G_B(1-\kappa))^\beta - \rho, \tag{102}$$

where $\hat{\pi}_A, \hat{\pi}_B, \hat{\pi}_{F\to A}, \hat{\pi}_{F\to B}$ are given in (12)-(13). Equation (101) is a nonlinear equation, but it has a unique solution. Indeed, as $\kappa \to 0$ or $\kappa \to 1$, the left-hand-side converges to $+\infty$ and $-\infty$, respectively, whereas the right-hand-side stays bounded. Furthermore, the left-hand-side is strictly decreasing in κ, while the right-hand-side is strictly increasing in κ. Notice that, all else equal, Δ_A affects the equilibrium by reducing the right-hand-side of equation (101). Consequently, κ is increasing in Δ_A as before – the reallocation effect. The leverage effect survives because the formula for μ^η is unchanged.

Third, the goods prices are equilibrium objects. The price ratio is given by

$$\frac{p_{A,t}}{p_{B,t}} = \frac{1-\beta}{\beta}\frac{G_B}{G_A}\frac{1-\kappa_t}{\kappa_t}. \tag{103}$$

This "exchange rate" allows an international economics interpretation. One could interpret sector A as domestic producers and sector B as foreign producers, with funds intermediated by a single global financial sector. The presence of κ_t in $p_{A,t}/p_{B,t}$ implies exchange rates are determined by global capital flows, unlike a frictionless complete-markets economy. As κ_t is influenced by financial variables like Δ_A, Δ_B and intermediary wealth η_t, global financial shocks affect exchange-rate dynamics, similar to the intermediary-centric theoretical analysis of Gabaix and Maggiori (2015). A diversification boom in one country can thus have spillovers to the global economy, through leverage increases in the global financial system.

E.2 Busts as Flight-to-Safety Episodes

Here, we consider an alternative model which also generates busts due to financiers' leverage constraint. Consider the economy from Section 3 without distressed investors (i.e., set $\chi = +\infty$) and without the overlapping generations structure (i.e., set $\delta = 0$). Introduce a new production technology as follows.

With $k \geq 0$ units of capital, this technology produces $\underline{r}k$, with $\underline{r} < \min(G_A, G_B)$. Despite being less productive, this technology is safer, in the sense that capital is riskless while being used as an input in this technology. There is a rental market for this capital, which must have rental rate $\underline{r}$, because of the linear production technology and absence of capital-quality shocks.

In this economy, the goods and bond market clearing conditions are modified to read

$$\int_0^1 [G_A k_{i,t}^A + G_B k_{i,t}^B + \underline{r}\,\underline{K}_t] di = \int_0^1 [c_{i,t}^A + c_{i,t}^B + c_{i,t}^F] di + \frac{1}{dt}\int_0^1 [dI_{i,t}^A + dI_{i,t}^B + d\underline{I}_{i,t}] di$$
$$\int_0^1 [n_{i,t}^A + n_{i,t}^B + n_{i,t}^F] di = \int_0^1 [k_{i,t}^A + k_{i,t}^B + \underline{K}_t] di,$$

where $\underline{K}_t$ and $d\underline{I}_t$ are the capital stock and investment flow into the riskless sector. As before, I will study a symmetric equilibrium, with state variables $K_t := \int_0^1 [k_{i,t}^A + k_{i,t}^B + \underline{K}_t] di$, $\eta_t := (\int_0^1 n_{i,t}^F di)/K_t$, and $\alpha_t := (\int_0^1 n_{i,t}^A di)/(\int_0^1 [n_{i,t}^A + n_{i,t}^B] di)$.

Using the definition of the aggregate investment rate ι_t and optimality conditions for consumption, we may re-write the goods market clearing condition, after scaling by K_t, as

$$\omega[\kappa G_A + (1-\kappa) G_B] + (1-\omega)\underline{r} = (1-\eta)\rho + \eta\rho_F + \iota. \tag{104}$$

In (104), ω is the capital share in the risky technologies A and B, whereas κ is the share, among the risky capital, in sector A. It is clear that, holding fixed η and κ, a decline in ω must reduce economic growth ι.

Finally, I allow any agent to access this technology, which implies $r_t \geq \underline{r}$ by absence of arbitrage. This implies the complementary-slackness condition

$$r_t \geq \underline{r}, \quad \omega_t \leq 1, \quad \text{and} \quad (1-\omega_t)(r_t - \underline{r}) = 0. \tag{105}$$

Because safe capital allocation implies lower production and growth, I will refer to times with $\omega_t < 1$ as periods of *misallocation*.

By repeating the arguments of Propositions C.3 and C.4, we may characterize the equilibrium. First, we find that equation (19) for ζ still holds. However, we must modify equation (69) for κ to account for the fact that risk prices are now scaled by the risky capital share ω. Indeed, insiders' and financiers' idiosyncratic risk prices are now given by

$$\omega\hat{\pi}_{F\to A}, \quad \omega\hat{\pi}_{F\to B}, \quad \omega\hat{\pi}_A, \quad \text{and} \quad \omega\hat{\pi}_B,$$

where $\hat{\pi}_{F\to A}, \hat{\pi}_{F\to B}, \hat{\pi}_A, \hat{\pi}_B$ are defined in (12) and (13). Similarly, aggregate risk prices are now given by

$$\omega\pi, \quad \omega\pi_A, \quad \text{and} \quad \omega\pi_B,$$

where π, π_A, π_B are defined in the statement of Proposition C.3. Equilibrium spreads are now given by

$$s_A - \omega\sigma_A \cdot \pi = \zeta + \omega(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}$$
$$s_B - \omega\sigma_B \cdot \pi = \zeta + \omega(1-\Delta_B)\hat{\sigma}_B\hat{\pi}_{F\to B}.$$

Using those definitions, we have the following equation for κ:

$$0 = \min\{1-\kappa,\, H^+\} - \min\{\kappa,\, (-H)^+\} \tag{106}$$
$$H := G_A - G_B - \phi_A s_A + \phi_B s_B - \omega(1-\phi_A)[\sigma_A\cdot\pi_A + \hat{\sigma}_A\hat{\pi}_A] + \omega(1-\phi_B)[\sigma_B\cdot\pi_B + \hat{\sigma}_B\hat{\pi}_B].$$

The equilibrium can be characterized by taking κ as given and solving for (ω, ζ, r). The result is in the following lemma.

Lemma E.1. *Equilibrium in the model with a positive-net-supply safe technology requires*

$$\omega = \min(1,\, \bar{\omega},\, \omega^*(0)) \tag{107}$$
$$\zeta = \mathbf{1}_{\{\bar{\omega}\leq 1\}} \max(0,\, \zeta^*) \tag{108}$$
$$r = \mathbf{1}_{\{\omega<1\}}\underline{r} + \mathbf{1}_{\{\omega=1\}} r^*(\zeta), \tag{109}$$

where

$$\bar{\omega} := (\kappa\phi_A + (1-\kappa)\phi_B)^{-1}\eta\bar{\lambda} \tag{110}$$
$$\omega^*(\zeta) := \frac{\kappa G_A + (1-\kappa)G_B - \underline{r} - \zeta(\kappa\phi_A + (1-\kappa)\phi_B)}{\eta(\|\pi\|^2 + \hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}) + (1-\eta)[\alpha(\|\pi_A\|^2 + \hat{\pi}^2_A) + (1-\alpha)(\|\pi_B\|^2 + \hat{\pi}^2_B)]} \tag{111}$$
$$\begin{aligned}\zeta^* := \frac{1}{\kappa\phi_A + (1-\kappa)\phi_B}\Big[&\kappa G_A + (1-\kappa)G_B - \underline{r} - \bar{\omega}\eta(\|\pi\|^2 + \hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}) \\ &- \bar{\omega}(1-\eta)(\alpha(\|\pi_A\|^2 + \hat{\pi}^2_A) + (1-\alpha)(\|\pi_B\|^2 + \hat{\pi}^2_B))\Big]\end{aligned} \tag{112}$$
$$\begin{aligned}r^*(\zeta) := {}&\kappa G_A + (1-\kappa)G_B - \zeta(\kappa\phi_A + (1-\kappa)\phi_B) \\ &- \eta(\|\pi\|^2 + \hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}) - (1-\eta)[\alpha(\|\pi_A\|^2 + \hat{\pi}^2_A) + (1-\alpha)(\|\pi_B\|^2 + \hat{\pi}^2_B)].\end{aligned} \tag{113}$$

Proof. When the leverage constraint in (19) binds, we may substitute financiers' optimal portfolios, along with equilibrium spreads, to get $\omega = \bar{\omega}$ from (110). At the same time, we may sum insiders' pricing conditions, the generalizations of (67) and (68) which account for ω, each weighted by κ and $1-\kappa$ respectively, to construct an equation $F(\omega,\zeta,r) = 0$ that holds in equilibrium. When $\omega = 1$, we have $F(1,\zeta,r) = 0$. This defines $r^*(\zeta)$ in (113). When $\omega < 1$, we have $r = \underline{r}$ by complementary-slackness condition (105), and so $F(\omega,\zeta,\underline{r}) = 0$. This defines $\omega^*(\zeta)$ in (111). Note that, by their definitions, $\omega^*(\zeta) \leq 1$ if and only if $r^*(\zeta) \leq \underline{r}$. Finally, define ζ^* by (112), which is the solution to $\bar{\omega} = \omega^*(\zeta)$.

Armed with these arguments, consider the following mutually-exclusive, completely-exhaustive cases:

- Suppose $\bar{\omega} > 1$. Then, $\eta^{-1}(\kappa\phi_A + (1-\kappa)\phi_B) < \bar{\lambda}$, so (19) implies $\zeta = 0$. Thus, equation (108) holds in this case. Consider the following sub-cases.
  - Suppose $\omega^*(0) \geq 1$. This implies $\omega = 1$ in which case $r = r^*(0)$. Thus, (107) and (109) hold.
  - Suppose $\omega^*(0) < 1$. This implies $\omega = \omega^*(0)$ in which case $r = \underline{r}$. Thus, (107) and (109) hold.
- Suppose $\bar{\omega} \leq 1$. Consider the following sub-cases.
  - Suppose $\omega^*(0) \geq \bar{\omega}$. Then, $\zeta^* \geq 0$. Hence, we may set $\omega = \bar{\omega}$, $\zeta = \zeta^*$, and $r = \underline{r}$ to satisfy (107), (108), and (109). To see that this is the unique option, notice that either $\omega = \omega^*(0)$ or $\omega = 1$ imply $\omega\eta^{-1}(\kappa\phi_A + (1-\kappa)\phi_B) \geq \bar{\lambda}$, which is a violation of (19) except in the trivial case $\omega = \bar{\omega}$.
  - Suppose $\bar{\omega} > \omega^*(0)$. Then, $\zeta^* < 0$. Hence, we are required to set $\omega = \omega^*(0)$, $\zeta = 0$, and $r = \underline{r}$ in order to satisfy (107), (108), and (109).

This completes the proof, since (107), (108), and (109) hold in all cases. □

To solve for equilibrium κ, we solve equation (106), using the results of Lemma E.1. This is a nonlinear equation, which must in general be solved numerically. In the case where sectors A and B are exactly symmetrical (i.e., $G_A = G_B$, $\phi_A = \phi_B$, $\hat{\sigma}_A = \hat{\sigma}_B$, $\|\sigma_A\| = \|\sigma_B\|$, and $\Delta_A = \Delta_B$), the equilibrium simplifies as the solution must be $\kappa = 1/2$ from (106). In this simple case, we can clearly illustrate the flight-to-safety episode induced by the presence of the leverage constraint.

Indeed, we have

$$\bar{\omega} < 1 \Leftrightarrow \eta < \eta_1^* := \phi/\bar{\lambda}$$

and

$$\bar{\omega} < \omega^*(0) \Leftrightarrow \eta < \eta_2^* := \frac{(G-\underline{r})\phi/\bar{\lambda} - \frac{1}{2}\phi^2(\|\sigma\|^2 + (1-\Delta)^2\hat{\sigma}^2)}{(G-\underline{r})\phi/\bar{\lambda} - \frac{1}{2}\phi^2(\|\sigma\|^2 + (1-\Delta)^2\hat{\sigma}^2) + (1-\phi)^2(\|\sigma\|^2 + \hat{\sigma}^2)}.$$

Thus, $\omega = \bar{\omega} < 1$ if and only if[46]

$$\eta < \eta_1^* \wedge \eta_2^*.$$

Thus, the "leverage effect" we have described in the main text can lead to real effects through misallocation. If diversification Δ increases enough, such that η_t drifts downwards, eventually financiers' leverage constraints will bind. Binding constraints imply misallocation in the sense that $\bar{\omega} < 1$. This is the notion of "endogenous bust" discussed in the main text. Furthermore, a subsequent negative aggregate shock lowers η_t even more, which reduces ω_t one-for-one, evidently from equation (110). This is the leverage-induced "instability" discussed in the main text. In summary, analogously to the main text, a model with the possibility of flight-to-safety can generate diversification-induced cycles which are both endogenous and unstable.

[46] Note that if

$$(\phi/\bar{\lambda})(1-\phi/\bar{\lambda})(G-\underline{r}) > (\phi/\bar{\lambda})(1-\phi)^2(\|\sigma\|^2+\hat{\sigma}^2) + (1-\phi/\bar{\lambda})\frac{\phi^2}{2}(\|\sigma\|^2+(1-\Delta)^2\hat{\sigma}^2),$$

then $\eta_1^* < \eta_2^*$, so that a binding leverage constraint and and misallocation are equivalent, i.e., they occur at exactly the same times. This result is analogous to part (ii) of Proposition 3.1.

F Empirical Analysis

F.1 Qualitative Support: Why the Model Applies to the US Housing Cycle

In addition to the *reallocation* and *leverage* patterns documented in figure 1, here I provide some more qualitative support for the mechanism of the model. First, the model requires that the increase in securitization actually improves diversification of mortgage loans. This is not necessarily true a priori: one possibility is that securitization of mortgage loans increases simply because the volume of mortgage lending increases. Figure 15 rejects this by showing that RMBS increase dramatically as a share of total household credit in the US. Moreover, non-agency MBS rise as a share of all MBS. Private label securitizations may be particularly important for diversification, because prior to the securitization boom, the types of loans in these pools were those most likely to be held on banks' balance sheets until maturity.

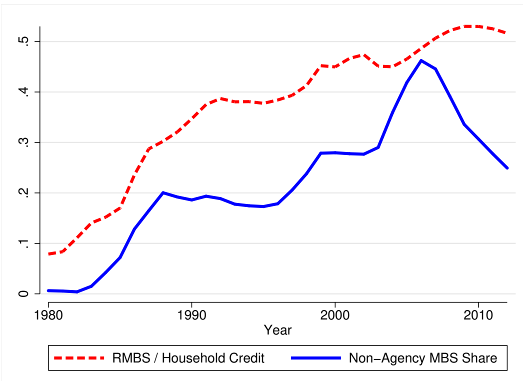

Figure 15: Securitization of household credit. "RMBS / Household Credit" sums both agency and non-agency RMBS and divides by the total household credit outstanding. "Non-Agency MBS Share" divides non-agency MBS by total MBS outstanding. Source: SIFMA and Flow of Funds.

Second, it is crucial for my results that diversification in the housing market increases more than diversification in the corporate credit market. This turns out to be true, if we measure diversification by securities, which are likely to be broadly held. Figure 16 shows that mortgage securities outstanding were equal to corporate securities outstanding in 1990, but nearly double by 2007.

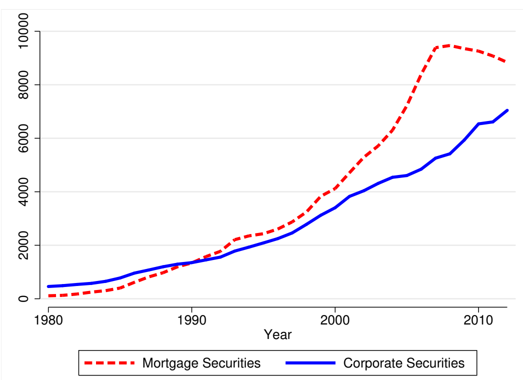

Figure 16: Mortgage versus Corporate Securities. "Mortgage Securities" are traded securities where the underlying assets are mortgages. "Corporate Securities" sums corporate bonds and any securitized bank loans. Source: SIFMA.

Third, my model assumes that the financial sector will adapt to an environment with better mortgage diversification by taking more housing-related risks onto their balance sheets. Figure 17 shows that commercial banks do indeed hold more housing-related assets on their balance sheets through the housing boom. Notice this series qualitatively mimics the household credit share from figure 1.

Similarly, figure 18 shows that price-to-cash-flow ratios in capital and housing markets do not move in lockstep, suggestive of some sectoral asymmetry in this boom period.

Finally, a key reason financial sector capitalization deteriorates in my model is through declining financier profitability. As diversification improves in the model, financiers are willing to accept lower risk premia on mortgages. Figure 19 shows that commercial banks' profitability declined marginally between the boom years 2000-2007.

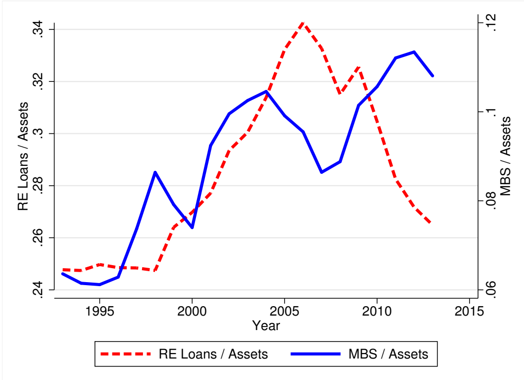

Figure 17: Commercial bank risk-taking in housing markets. “RE Loans / Assets” refers to real estate loans held on bank balance sheets, relative by assets. “MBS / Assets” are mortgage-backed securities held, relative to assets. Source: Call Reports.

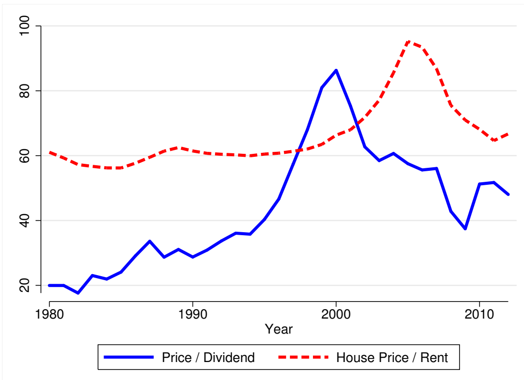

Figure 18: The price-dividend ratio on the S&P 500 and a measure of house prices relative to the rental rate on housing services. The house price-rent ratio is obtained from http://datatoolkits.lincolninst.edu/subcenters/land-values. The plotted ratio is scaled by 3.

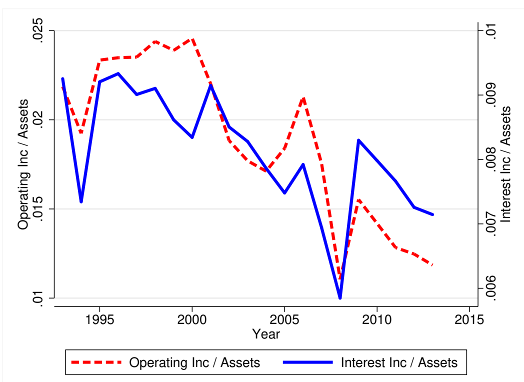

Figure 19: Commercial bank profitability. “Operating Inc / Assets” is operating income, relative to assets. “Interest Inc / Assets” is income from interest payments, relative to assets. Source: Call Reports.

F.2 Quantifying Mortgage Diversification

In this appendix, I describe more specifically the methodology to compute the diversification index of Section 4.1. Start by defining an aggregate mortgage return during month k of year t:

$$\overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} := \sum_{\ell} \omega_{\ell,t} R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}$$

where $\omega_{\ell,t} := \frac{s_{\ell,t}+m_{\ell,t}}{\sum_{\ell'} s_{\ell',t}+m_{\ell',t}}$ are origination weights:

$$\begin{aligned} m_{\ell,t} &:= \text{ portfolio mortgages originated to location } \ell \text{ in year } t \\ s_{\ell,t} &:= \text{ securitized mortgages originated to location } \ell \text{ in year } t. \end{aligned}$$

The location-specific mortgage return $R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}$ is proxied by the housing return in location ℓ and month k of year t, taken from CoreLogic. This is the return building block for all other returns. The aggregate return $\overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}$ allows me to extract the idiosyncratic components of all other returns.

In an analogous fashion, define the mortgage return for intermediary i:

$$R^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} := \sum_{\ell} \omega^{(i)}_{m,\ell,t} R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}} + \omega^{(i)}_{s,\ell,t} \overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}$$

where

$$\omega^{(i)}_{m,\ell,t} := \frac{m^{(i)}_{\ell,t}}{\sum_{\ell'} s^{(i)}_{\ell',t} + m^{(i)}_{\ell',t}} \quad \text{and} \quad \omega^{(i)}_{s,\ell,t} := \frac{s^{(i)}_{\ell,t}}{\sum_{\ell'} s^{(i)}_{\ell',t} + m^{(i)}_{\ell',t}}$$

$$m^{(i)}_{\ell,t} := \text{ portfolio mortgages originated by lender } i \text{ to location } \ell \text{ in year } t$$

$$s^{(i)}_{\ell,t} := \text{ sold mortgages originated by lender } i \text{ to location } \ell \text{ in year } t.$$

Note that any mortgages originated by intermediary i which are then sold off within the same year are captured by $s^{(i)}_{\ell,t}$. I make the assumption that these sales return the aggregate return, which is subject to no idiosyncratic risk. Loans not sold are captured by $m^{(i)}_{\ell,t}$. I apply the location-specific return to these loans.

Next, I define "idiosyncratic returns" by subtracting the aggregate return:

$$\mathcal{R}_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}} := R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}} - \overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}$$

$$\mathcal{R}^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} := R^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} - \overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}$$

The (monthly) idiosyncratic variances in year t are then given by

$$\mathcal{V}^2_{\ell,t} := \frac{1}{12}\sum_{k=1}^{12}\left(\mathcal{R}_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2 - \left(\frac{1}{12}\sum_{k=1}^{12}\mathcal{R}_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2$$

$$\mathcal{V}^2_{i,t} := \frac{1}{12}\sum_{k=1}^{12}\left(\mathcal{R}^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2 - \left(\frac{1}{12}\sum_{k=1}^{12}\mathcal{R}^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2$$

In this computation, I am using the fact that the returns are computed monthly, while the originations and securitizations data are only available at a yearly frequency.

I average over locations and intermediaries to get the volatilities that I want:

$$\hat{\sigma}_t := \sum_{\ell} \omega_{\ell,t} \mathcal{V}_{\ell,t}$$

$$\hat{\sigma}_{\Delta,t} := \sum_{i} \omega_{i,t} \mathcal{V}_{i,t}$$

where

$$\omega_{i,t} := \sum_{\ell} \frac{m^{(i)}_{\ell,t} + s^{(i)}_{\ell,t}}{\sum_{\ell',j} m^{(j)}_{\ell',t} + s^{(j)}_{\ell',t}}.$$

Note that we necessarily have $\hat{\sigma}_{\Delta,t} \leq \hat{\sigma}_t$, because correlations between the loans in lender's portfolios are less than 1, while loan-level volatilities are proxied by location-specific volatilities.

Finally, in the symmetric equilibrium, the following equation relates financiers' housing risk $\hat{\sigma}_{\Delta}$ to the fundamental housing risk $\hat{\sigma}$ and the level of diversification Δ:

$$(1-\Delta)\hat{\sigma} = \hat{\sigma}_{\Delta} = \text{ idio volatility of unlevered mortgage portfolio.}$$

Thus, by inverting this relation, I define

$$\Delta_t := 1 - \hat{\sigma}_{\Delta,t}/\hat{\sigma}_t.$$

Units of Δ_t are the fraction of fundamental housing risk that are eliminated from lender's portfolios, either through loan sales and securitizations, or through geographic diversification.