Financial Frictions and Aggregate Fluctuations*

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Abstract

Concentrated idiosyncratic risk positions may generate aggregate fluctuations. I study a canonical macroeconomic model with a standard moral hazard friction but with a single innovation: fundamental shocks are correlated (but still aggregate to zero). Experts hold concentrated asset positions, while less productive households hold diversified positions in experts' equity. I prove that aggregate output and the wealth distribution feature aggregate volatility if and only if observability and contractibility are imperfect. This failure of the law of large numbers holds generically and does not require any assumptions about fat-tailed size or wealth distributions. Even though aggregate volatility disappears with perfect contractibility, it can increase with partial contractibility improvements, due to market segmentation between experts and households. These results are immune to allowing agents to frictionlessly hedge the endogenously-arising aggregate shocks.

JEL Codes: D14, G11, G12.

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1 Introduction

Which shocks are the source of macroeconomic fluctuations? Much of the macroeconomic literature points to aggregate shocks affecting representative firms or representative consumers. Some example candidates are total factor productivity shocks, investment-specific shocks, intertemporal preference shocks, labor supply shocks, price mark-up shocks, government spending shocks, and monetary policy shocks. These are the shocks used, for example, in the seminal work of Smets and Wouters (2007). In the aftermath of the 2008 financial crisis, recent research has focused on the importance of financial or uncertainty shocks, e.g., Jermann and Quadrini (2012) and Christiano, Motto and Rostagno (2014).

This paper joins a different literature that views *aggregation failure* as an important source of business cycle fluctuations. In other words, firms or consumers are subject to idiosyncratic shocks that do not "average out" in the aggregate. The motivation for doing this stems from the direct evidence available on micro-level shocks and lack of direct, non-structural evidence on macro-level shocks.

Non-aggregation can occur for a variety of reasons. For example, if there is a fat-tailed size distribution, some idiosyncratic shocks are weighed heavily in aggregates (e.g., Gabaix (2011)). Alternatively, highly concentrated network linkages can lead to extreme shock propagation, which translates into aggregate volatility (e.g., Carvalho (2010) and Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012)).[1]

In this paper, I adopt a blend of these two philosophies. I model a continuum of correlated but idiosyncratic shocks and think about aggregating these shocks against an endogenous wealth distribution. The shocks are idiosyncratic microeconomic shocks, in the sense that they aggregate to zero when weighed against the unit kernel. This kernel represents the distribution of sizes or wealths, and this will be a key object in determining aggregate volatility, as in the "granularity" hypothesis of Gabaix (2011). But the shocks are correlated with each other, which is a reduced-form assumption meant to capture the possibility of an underlying network of interlinkages, as in Acemoglu et al. (2012).

In the presence of such correlated idiosyncratic shocks, I prove a technical result driving all my main findings: for (almost) any non-unit kernel, the shocks do not aggregate. In words, for any distribution of sizes or wealths besides the degenerate symmetric distribution, idiosyncratic shocks translate into aggregate fluctuations, which are the kernel-weighted sum of the idiosyncratic shocks. Because of the assumed shock correlation structure, my economy features a non-aggregation result that does not require fat-tailed size distributions ("granularity") or highly-concentrated production networks with critical nodes

[1] Other mechanisms are explored in the papers of Jovanovic (1987), Durlauf (1993), and Nirei (2006).

("network hubs").

Economically, the non-degeneracy of the wealth distribution arises if and only if financial frictions are present. Due to moral hazard considerations, agents must bear some fraction ϕ of the idiosyncratic risk embodied in the capital that they manage. This parameter represents the "skin in the game" required for incentive provision. Because of this non-diversifiability, the wealth distribution across agents cannot evolve symmetrically, even if the economy starts with equally-distributed wealth. Thus, the entire wealth distribution across agents becomes a state variable in dynamic equilibrium. This dynamic drifting apart generates positive volatility of aggregate variables, due to the technical non-aggregation result discussed above. Conversely, without moral hazard, or with perfect observability and contractibility (such that $\phi = 0$), the equilibrium wealth distribution becomes symmetric and aggregate volatility vanishes.

My paper features a failure of the law of large numbers, which is a difference relative to most of the extant literature. For example, in the granularity hypothesis of Gabaix (2011), a law of large numbers still holds as the number of firms $N \to \infty$, but the standard central limit theory does not hold.[2] The network papers of Carvalho (2010) and Acemoglu et al. (2012) also have a law of large numbers under most network structures.[3] Those papers consider the rate of decay in aggregate volatility as N gets large. In my model, even a continuum of idiosyncratic shocks do not wash out in the aggregate.

To illustrate this difference, consider the following series of examples. In all of them, there are N firms with sizes ω_i, with average size $N^{-1}\sum_{i=1}^{N}\omega_i = 1$. Each firm draws a shock ε_i having volatility σ. Consider aggregate output growth $g := N^{-1}\sum_{i=1}^{N}\omega_i\varepsilon_i$.

With iid shocks and equal sizes ($\omega_i = 1$ for all i), the standard deviation of g is $\sigma/N^{1/2}$. This fast decay of risk corresponds to the standard diversification arguments given in the traditional macroeconomics literature.[4]

Now, suppose the size distribution if fat-tailed, in the following sense. The first firm has $\omega_1 = \rho N^{\alpha}$, with $0 < \alpha, \rho < 1$, so that its absolute size grows with the size of the economy, but its relative size becomes negligible. Suppose the other firms are equal-sized, so $\omega_i = (N - \omega_1)/(N-1)$ for all $i \in \{2, \dots, N\}$. Then the standard deviation of g is $(\sigma/(N-1)^{1/2})\big[1 - 2\rho N^{\alpha-1} + N(\rho N^{\alpha-1})^2\big]^{1/2}$, which asymptotically behaves like $\sigma/N^{1/2} + \rho\sigma/N^{1-\alpha}$. The second term, which can decay very slowly for α close to 1, arises due to the non-trivial

[2] This is also true for the papers cited above: Jovanovic (1987), Durlauf (1993), Nirei (2006).

[3] The prominent counterexample from those papers is the "star network" in which one firm is a supplier to every other firm in the economy. In such a case, aggregate volatility does not vanish as N gets large because the influence of that central supplier firm grows with the size of the economy.

[4] See, more generally, the irrelevance theorems of Dupor (1999), which arrive at the same diversification conclusion even under input-output linkages.

size of firm 1. This is the "granularity" argument of Gabaix (2011).

Next, suppose again that all firms are equal-sized ($\omega_i = 1$ for all i) but that the shocks have some correlation. Indeed, let the shocks of all firms $i \neq 1$ depend on firm 1 via $\varepsilon_i = \rho N^{\alpha-1}\varepsilon_1 + \sqrt{1-(\rho N^{\alpha-1})^2}\tilde{\varepsilon}_i$, where $0 < \alpha, \rho < 1$ and $\{\tilde{\varepsilon}_i\}_{i\neq 1}$ are iid copies of ε_1. Such strong dependence might arise endogenously if the first firm is a key supplier to all other firms in the economy, as in a "star network." In this model, the standard deviation of g is $(\sigma/N^{1/2})\big[1 + \frac{N-1}{N}(\rho N^{\alpha-1}) + \frac{N-1}{N}(N-2)(\rho N^{\alpha-1})^2\big]^{1/2}$, which asymptotically behaves like $\sigma/N^{1/2} + \rho\sigma/N^{1-\alpha}$. In structural network models like Acemoglu et al. (2012), the deep shocks are assumed independent, but linkages lead to behavior that looks *as if* the shocks were correlated. This reduced-form example shows how such correlated shocks can lead to slow decay in aggregate volatility.

My paper combines these arguments – correlated shocks and a non-trivial size distribution – thus leading to a stronger form of non-aggregation than each of these theories individually. The starting point of my paper is a reduced-form assumption that shocks are correlated, but still idiosyncratic in the sense that Var$(N^{-1}\sum_{i=1}^N \varepsilon_i) \to 0$ as $N \to \infty$. Over time, due to financial frictions in the economy, the size distribution $\{\omega_i\}_{i=1}^N$ endogenously deviates from the symmetric distribution. My key technical result, under these assumptions, is that Var$(N^{-1}\sum_{i=1}^N \omega_i\varepsilon_i) \to 0$ if and only if $\omega_i = 1$ for (essentially) all i. This technical lemma drives the non-aggregation result in this paper.[5]

Relative to the network and granularity literatures, the relatively stronger non-aggregation results of this paper can emerge out of relatively weaker primitive conditions. My shock dependence structure features less prominent linkages than the "star network" example above. My size distribution does not need to be fat-tailed as in the example above. By combining insights, my paper can weaken the premises of the network and granularity arguments.

The remainder of the paper is organized as follows. Section 2 develops the correlated shock structure and the economic environment. Section 3 then derives the equilibrium in closed-form. Section 4 analyzes the equilibrium, focusing on the effect of financial frictions on aggregate fluctuations. Section 5 concludes. Proofs are in Appendix A, unless indicated otherwise.

2 Model Setup

Time is continuous $t \geq 0$. The model features two groups of agents: experts (E) and households (H). Agents in each group are additionally indexed by $i \in [0,1]$, which will represent

[5] I work directly in the continuum limit $i \in [0,1]$ rather than directly taking $N \to \infty$.

an agent's location, to be described below. Experts invest in and manage capital. Because of their higher productivity, they will typically be levered, borrowing from households in the riskless bond market. Households are less productive at managing capital, and will typically abstain from production, instead holding fully diversified positions in expert outside equity as well as riskless savings.

Preferences

Experts and households are infinitely-lived and have logarithmic utility over the single non-durable consumption good (the numeraire). Mathematically,

$$\mathcal{U}_t := \mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(s-t)}\log(c_s)ds\Big], \quad \rho > 0. \tag{1}$$

I assume experts and households have potentially different subjective discount rates ρ_E and ρ_H.

Locations and Idiosyncratic Risk

Agents are arranged on a circle, which has locations indexed by $i \in [0,1]$. Locations will be special because they feature different shocks. The technical results of this section borrow heavily from Khorrami (2018). All proofs of claims in this section are in Appendix B of that paper.

Each location i has its own productive capital stock $k_{i,t}$. The consumption good is produced in every location by the capital stock of that location, but the good is freely tradable across locations. Similarly, capital stocks are tradable within and across locations.

Over time, shocks directly hit the evolution of $k_{i,t}$. Capital held by an agent at location i evolves dynamically as

$$dk_{i,t} = k_{i,t}[\iota_{i,t}dt + \sigma dW_{i,t}], \tag{2}$$

where $\iota_{i,t}$ is the desired investment rate and $W_{i,t}$ is the location-specific shock (more on this below). These "capital-quality shocks" $\sigma dW_{i,t}$ are a simple way to capture permanent productivity shocks, without introducing additional state variables. Investment is not subject to any adjustment costs.

I assume the idiosyncratic shock $W_{i,t}$ has the following properties.

Assumption 1 (Shock Structure)**.** *Assume the following for* $W := \{W_{i,t} : i \in [0,1], t \geq 0\}$.

(i) At each location $i \in [0,1]$, $W_{i,t}$ *is a standard Brownian motion, independent of* Z_t.

(ii) For any two locations $i, j \in [0,1]$, the shock correlation is

$$\text{corr}(dW_{i,t}, dW_{j,t}) = 1 - 6\text{dist}(i,j)(1 - \text{dist}(i,j)),$$

where $\text{dist}(i,j) := \min(|i-j|, 1-|i-j|)$ *is the distance on a circle of circumference 1.*

(iii) $W_{i,t}$ *is continuous in* (i,t) *almost-surely, under the Euclidean distance metric on the cylinder, i.e.,* $\widetilde{\text{dist}}((i,s),(j,t)) := [|s-t|^2 + \text{dist}(i,j)^2]^{1/2}$.

Given part (i) of Assumption 1, $dW_{i,t}$ is iid over time, for fixed location i. Part (ii) of Assumption 1 means that the shock correlations between locations decrease with their distance from one another.[6] Nearer locations have higher shock correlations than further locations. I will call W a *Brownian cylinder*, because it evolves on a circle over time, which looks like a cylinder. A key question is whether any such stochastic process exists.

Lemma 2.1. *The Brownian cylinder exists on some probability space.*

With the properties in Assumption 1, we can establish that the Brownian cylinder W contributes no aggregate risk.

Lemma 2.2. *Under Assumption 1,* $\int_0^1 (dW_{i,t})di = 0$ *almost-surely.*[7]

Given Lemma 2.2, the shock $dW_{i,t}$ is correlated across locations but washes out in the aggregate, the sense in which it is idiosyncratic. However, this aggregation result is fragile. This is the content of the following lemma, which is the linchpin in all of the equilibrium results about aggregate fluctuations arising from idiosyncratic shocks.

Lemma 2.3. *Let* $\{\omega_{i,t} : i \in [0,1], t \geq 0\}$ *be a positive stochastic process satisfying* $\int_0^1 \omega_{i,t} di = 1$ *for all* t *and continuous in* (i,t) *almost-surely. Then,* $\text{Var}_t(\int_0^1 \omega_{i,t} dW_{i,t} di) = 0$ *if and only if* $\omega_{i,t} = 1$ *for almost every* i.

Lemma 2.3 says that, if there exists any positive measure subset of locations having a non-degenerate size distribution, the aggregation of the Brownian cylinder against this size kernel fails. This is important for my results because my model generates a non-degenerate distribution of wealth over time. This will be the source of aggregate fluctuations.

[6]This presumption on the shock correlation owes to Gârleanu, Panageas and Yu (2015). Using the Brownian bridge on a "circle," they show how to construct discrete-time idiosyncratic shocks that are cross-sectionally correlated but contain zero aggregate risk. In the process, they find that the dividend correlation is exactly $1 - 6\text{dist}(i,j)(1 - \text{dist}(i,j))$. The proof that there exists a process satisfying Assumption 1, Lemma 2.1, would apply for any appropriate correlation function $v(i,j)$ that depends only on $\text{dist}(i,j)$ (i.e., stationary correlation function).

[7]The notation "$dW_{i,t}di$" is shorthand for the correct notation "$W(di,dt)$" where W is a measure on $[0,1] \times \mathbb{R}$. However, given the multiplicatively separable correlation structure in Assumption 1, writing "$dW_{i,t}di$" is also valid.

Return-on-Capital

A firm is just a collection of capital, which produces according to an "AK" technology. I assume experts are more productive than households, so $a_E > a_H$. With zero capital adjustment costs, the unit price of capital is always equal to one, so there is a very simple return on capital:

$$dR^g_{i,t} = a_g dt + \sigma dW_{i,t}, \quad g \in \{E, H\}. \tag{3}$$

Constant expected returns makes the analysis easier.

Outside Equity

Any agent (expert or household) that manages capital may partially finance their purchases by issuing outside equity. Issuance is a way for an insider (whether that be an expert or household) to shed some of the idiosyncratic risk associated with their capital holdings.

For simplicity, I assume that an insider issues a fixed fraction $1 - \phi$ of their capital holdings as outside equity and retains ϕ fraction as inside equity. With the fixed fraction, insiders pay $(1 - \phi)d\tilde{R}_{i,t}$ per unit of capital to the equity market, where $d\tilde{R}_{i,t}$ is the return on outside equity (to be described below). Although I specify this friction exogenously, such a fixed risk-sharing arrangement can be derived as the optimal short-term contract in a standard moral hazard problem, as in Di Tella (2017). See also Appendix A of Khorrami (2018). When $\phi = 0$, there are no moral hazard frictions, which corresponds to perfect observability and contractibility.[8]

The outside equity return is given by

$$d\tilde{R}_{i,t} := (r_t + \gamma_t)dt + \sigma dW_{i,t}. \tag{4}$$

A discussion of equation (4) is in order. First, notice that $d\tilde{R}_{i,t}$ has the same risk loadings as $dR^E_{i,t}$ and $dR^H_{i,t}$, which is why I have called this contract equity. Second, the expected excess return required is given by γ_t, which is independent of location i, because I assume that anytime experts or households access equity markets, they trade the complete market portfolio of equity.

[8] Under the moral hazard interpretation, the restriction that experts keep ϕ fraction of capital risk on their balance sheets is called a "skin-in-the-game" constraint. Given sufficient skin in the game, the exact composition of outside contracts is irrelevant. Indeed, once moral hazard problems are resolved between insiders and outsiders, the outside securities issued by households are indeterminate due to Modigliani-Miller holding on these securities. In particular, there are no taxes, costs of default, incomplete financial markets, or any other frictions that would violate MM, after agency problems are resolved. Therefore, the equity-like contract is without loss of generality under this interpretation.

Indeed, I do not allow an insider unrestricted location-specific trading positions in outside equity. Allowing this would remove all agency constraints. For example, an expert in location i could short the equity of the expert in location $i + \varepsilon$, for some ε arbitrarily small. Since the correlations between location i and $i + \varepsilon$ converge monotonically to 1 as $\varepsilon \to 0$, such a strategy can dissolve expert i's idiosyncratic risk. Thus, agency considerations would typically restrict the trading of equity securities to a diversified portfolio.[9]

This diversified portfolio pays

$$d\tilde{R}_t^* = (r_t + \gamma_t)dt + \sigma dZ_{R,t}^*, \tag{5}$$

where

$$dZ_{R,t}^* = \int_0^1 \lambda_{i,t} dW_{i,t}, \quad \int_0^1 \lambda_{i,t} di = 1, \tag{6}$$

and $\lambda_{i,t}$ is to be determined in equilibrium. In standard models with iid shocks, the idiosyncratic risks would wash out in the aggregate, so $Z_{R,t}^* = 0$ almost-surely. In such a case, $d\tilde{R}_t$ is locally deterministic, and arbitrage dictates that $\gamma_t = 0$ as well. But $\lambda_{i,t} \neq 1$ on a positive measure subset of $[0, 1]$, as I will show in equilibrium. According to Lemma 2.3, this causes aggregation to fail, and $dZ_{R,t}^*$ survives as an aggregate shock.

Expert Problem

Experts are the high-productivity agents in the economy. Some examples might be specialists in production and innovation, entrepreneurs, and certain types of financial professionals. On the asset side, experts hold capital that returns (3). On the liability side, experts are financed by their own net worth as well as outside equity that pays (4). Experts are also marginal in the risk-free bond market, at the interest rate r_t. Finally, experts are allowed to hold unrestricted positions in the market portfolio of outside equity.

[9] For an agent that chooses to hold no capital, holding the market equity portfolio is optimal. Indeed, agents have symmetric risk preferences, their portfolio problems are scale-invariant, and the probability distribution of asset returns (capital, outside equity) is location-invariant. Using these facts, one can show that all agents within a group (experts, households) hold positive quantities of capital or zero, regardless of location. When households do hold capital, agency considerations would typically restrict them from taking completely unrestricted positions in other locations' equity, as with experts. Please see the discussion above.

Combining the assumptions above, expert net worth $n^E_{i,t}$ evolves as

$$dn^E_{i,t} = \underbrace{(n^E_{i,t} r_t - c^E_{i,t})dt}_{\text{consumption-savings}} + \underbrace{k^E_{i,t}(dR^E_{i,t} - r_t dt)}_{\text{capital holdings}} - \underbrace{(1-\phi)k^E_{i,t}(d\tilde{R}_{i,t} - r_t dt)}_{\text{outside equity}} + \underbrace{\theta^E_{i,t} n^E_{i,t}(d\tilde{R}^*_t - r_t dt)}_{\text{market indeces}}, \tag{7}$$

Mathematically, experts solve

$$\max_{n^E_i, c^E_i, k^E_i} \mathcal{U}^E_{i,t} \tag{8}$$

subject to (7), $n^E_{i,t} \geq 0$, $k^E_{i,t} \geq 0$, where $\mathcal{U}^E_{i,t}$ is given by the logarithmic utility function (1).

Household Problem

Households are the same as experts in every way, except they have a lower productivity when managing capital. Thus, the household's net worth evolves dynamically as follows:

$$dn^H_{i,t} = \underbrace{(n^H_{i,t} r_t - c^H_{i,t})dt}_{\text{consumption-savings}} + \underbrace{k^H_{i,t}(dR^H_{i,t} - r_t dt)}_{\text{capital holdings}} - \underbrace{(1-\phi)k^H_{i,t}(d\tilde{R}_{i,t} - r_t dt)}_{\text{outside equity}} + \underbrace{\theta^H_{i,t} n^H_{i,t}(d\tilde{R}^*_t - r_t dt)}_{\text{market indeces}}. \tag{9}$$

Households solve

$$\max_{n^H_i, c^H_i, k^H_i} \mathcal{U}^H_{i,t} \tag{10}$$

subject to (9), $n^H_{i,t} \geq 0$, $k^H_{i,t} \geq 0$, where $\mathcal{U}^H_{i,t}$ is given by the logarithmic utility function (1).

Limited Mobility

To generate aggregate fluctuations, I must prohibit net worth from flowing frictionlessly across locations, in such a way as to equalize capital and wealth distributions. I make an extreme version of this assumption below.

Assumption 2 (Limited Mobility)**.** *Experts and households cannot choose to move across locations i. At Poisson rate $\delta > 0$, agents independently move to another location i', which is chosen at random (according to the uniform distribution).*

Under Assumption 2, idiosyncratic shocks will not wash out in aggregate. Khorrami (2018) considers the polar opposite case where agents may move costlessly across locations.

In that case, it is possible to study a symmetric equilibrium, in which idiosyncratic shocks do wash out in the aggregate.

3 Equilibrium

Definition 1. *An* equilibrium *consists of price and allocation processes, adapted to the aggregate and idiosyncratic shocks* $\{(Z_t, W_{i,t}) : i \in [0,1], t \geq 0\}$*, such that all agents solve their optimization problems and all markets clear. These objects consist of the interest rate* r_t*, the spread* γ_t*, capital holdings* $(k^E_{i,t}, k^H_{i,t})$*, and consumption choices* $(c^E_{i,t}, c^H_{i,t})$*. The market clearing conditions at every point in time are as follows.*

- Goods market:

$$\int_0^1 [a_E - \iota^E_{i,t}] k^H_{i,t} di + \int_0^1 [a_H - \iota^H_{i,t}] k^H_{i,t} di = \int_0^1 [c^E_{i,t} + c^H_{i,t}] di.$$

- Bond market:

$$\int_0^1 [n^E_{i,t} + n^H_{i,t}] di = \int_0^1 [k^E_{i,t} + k^H_{i,t}] di.$$

- Equity market:

$$\int_0^1 [\theta^E_{i,t} n^E_{i,t} + \theta^H_{i,t} n^H_{i,t}] di = (1-\phi) \int_0^1 [k^E_{i,t} + k^H_{i,t}] di.$$

- Capital market:

$$k^E_{i,t} + k^H_{i,t} = k_{i,t}, \quad i \in [0,1].$$

We proceed to construct an equilibrium. In short, because agents have log preferences and scale-invariant portfolio choice problems, the equilibrium can be obtained in closed-form. This is despite the presence of financial and mobility frictions.

To start, define the aggregates

$$K_{E,t} := \int_0^1 k^E_{i,t} di \quad \text{and} \quad K_{H,t} := \int_0^1 k^H_{i,t} di$$
$$N_{E,t} := \int_0^1 n^E_{i,t} di \quad \text{and} \quad N_{H,t} := \int_0^1 n^H_{i,t} di.$$

Because of scale-invariance of this economy, several equilibrium objects will scale with the levels of total capital $K_t := K_{E,t} + K_{H,t}$ and total wealth $N_{E,t} + N_{H,t}$. As such, let x_t and η_t

be defined by experts' aggregated capital and wealth shares:

$$x_t := K_{E,t}/(K_{E,t} + K_{H,t})$$
$$\eta_t := N_{E,t}/(N_{E,t} + N_{H,t}).$$

In addition, description of the equilibrium requires variables pertaining to the *distributions* of capital and wealth among experts and households:

$$\omega_{i,t} := k_{i,t}/K_t$$
$$\eta^E_{i,t} := n^E_{i,t}/N_{E,t}$$
$$\eta^H_{i,t} := n^H_{i,t}/N_{H,t}.$$

The wealth share variables $(\eta_t, \{\eta^E_{i,t}, \eta^H_{i,t} : i \in [0,1]\})_{t\geq 0}$ are state variables for this economy, whereas the capital share variables $(x_t, \{\omega_{i,t} : i \in [0,1]\}_{t\geq 0}$ are determined based on the wealth shares, because capital holdings are choice variables for individual agents. Furthermore, because these capital choice decisions are homogeneous in agents' wealth, the aggregate capital share x_t can be determined simply from the aggregate wealth share η_t. By contrast, the capital distribution across locations is determined by the wealth distribution across locations. These results are collected in the following lemma.

Lemma 3.1. *Let $\phi > 0$, and let Assumptions 1 and 2 hold. Define expert and household* idiosyncratic risk prices

$$\pi_{E,t} := \frac{a_E - r_t - (1-\phi)\gamma_t}{\phi\sigma} \quad \textit{and} \quad \pi_{H,t} := \Big(\frac{a_H - r_t - (1-\phi)\gamma_t}{\phi\sigma}\Big)^+. \tag{11}$$

In equilibrium,

$$\pi_{E,t} = \min\Big(\phi\sigma + \frac{(1-\eta_t)(a_E - a_H)}{\phi\sigma}, \frac{\phi\sigma}{\eta_t}\Big) \tag{12}$$
$$\pi_{H,t} = \max\Big(\phi\sigma - \frac{\eta_t(a_E - a_H)}{\phi\sigma}, 0\Big). \tag{13}$$

The aggregate capital distribution $x_t := K_{E,t}/(K_{E,t} + K_{H,t})$ is given by

$$x_t = \min\Big(\eta_t\Big[1 + \frac{(1-\eta_t)(a_E - a_H)}{\phi^2\sigma^2}\Big], 1\Big). \tag{14}$$

The location-specific capital distribution $\omega_{i,t} := k_{i,t}/K_t$ is given by

$$\omega_{i,t} = \frac{1}{\phi\sigma}\Big[\eta_t \eta^E_{i,t}\pi_{E,t} + (1-\eta_t)\eta^H_{i,t}\pi_{H,t}\Big], \quad \forall i \in [0,1]. \tag{15}$$

The following two lemmas characterize the evolutions of the state variables. To do this, we must define the following aggregates of the Brownian cylinder, weighted against these endogenous capital and wealth distributions:

$$\begin{aligned} dZ^*_{K,t} &:= \int_0^1 \omega_{i,t} dW_{i,t} di \\ dZ^*_{E,t} &:= \int_0^1 \eta^E_{i,t} dW_{i,t} di \\ dZ^*_{H,t} &:= \int_0^1 \eta^H_{i,t} dW_{i,t} di. \end{aligned}$$

These "shocks" arise endogenously and are non-zero due to the unequal capital and wealth shares, as we will see shortly.

Lemma 3.2. *Aggregate capital evolves as*

$$dK_t = K_t\Big[x_t a_E + (1-x_t)a_H - \eta_t\rho_E - (1-\eta_t)\rho_H\Big]dt + K_t\sigma dZ^*_{K,t}. \tag{16}$$

Note that $dZ^*_{K,t} = dZ^*_{R,t}$. Thus, the aggregate risk premium is given by

$$\gamma_t = (1-\phi)\sigma^2 z_t^2, \tag{17}$$

where $z_t^2 dt := \mathit{Var}_t(dZ^*_{K,t})$.

Lemma 3.3. *The aggregate wealth share η_t evolves as*

$$\begin{aligned} d\eta_t = \eta_t(1-\eta_t)&\Big[\rho_H - \rho_E + \pi^2_{E,t} - \pi^2_{H,t} + \eta_t\bar{\Psi}_{E,t} - (1-\eta_t)\bar{\Psi}_{H,t} + (1-2\eta_t)\bar{\Psi}_{E,H,t}\Big]dt \\ &+ \eta_t(1-\eta_t)\Big[\pi_{E,t}dZ^*_{E,t} - \pi_{H,t}dZ^*_{H,t}\Big]. \end{aligned} \tag{18}$$

where

$$\begin{aligned} \bar{\Psi}_{E,t}dt &:= \mathit{Var}_t(d\log N_{E,t}) \\ \bar{\Psi}_{H,t}dt &:= \mathit{Var}_t(d\log N_{H,t}) \\ \bar{\Psi}_{E,H,t}dt &:= \mathit{Cov}_t(d\log N_{E,t}, d\log N_{H,t}) \end{aligned}$$

are quadratic variations of aggregated wealths. The location-specific wealth shares $(\eta^E_{i,t}, \eta^H_{i,t})$ *evolve as*

$$d\eta^E_{i,t} = \delta(\eta^E_{i,t})^{-1}(1-\eta^E_{i,t})dt + \eta^E_{i,t}\pi_{E,t}\Big[\bar{\Phi}_{E,t} - \Phi_{E,i,t}\Big]dt + \eta^E_{i,t}\pi_{E,t}\Big[dW_{i,t} - dZ^*_{E,t}\Big] \tag{19}$$

$$d\eta^H_{i,t} = \delta(\eta^H_{i,t})^{-1}(1-\eta^H_{i,t})dt + \eta^H_{i,t}\pi_{H,t}\Big[\bar{\Phi}_{H,t} - \Phi_{H,i,t}\Big]dt + \eta^H_{i,t}\pi_{H,t}\Big[dW_{i,t} - dZ^*_{H,t}\Big], \tag{20}$$

where

$$\Phi_{E,i,t}dt := \mathit{Cov}_t(dW_{i,t}, d\log N_{E,t})$$
$$\Phi_{H,i,t}dt := \mathit{Cov}_t(dW_{i,t}, d\log N_{H,t})$$

are quadratic covariations between wealth and the idiosyncratic shocks, and

$$\bar{\Phi}_{E,t} := \int_0^1 \Phi_{E,i,t}di \quad \textit{and} \quad \bar{\Phi}_{H,t} := \int_0^1 \Phi_{H,i,t}di$$

aggregate these quadratic covariations.

Theorem 3.4. *Let $\phi > 0$, and let Assumptions 1 and 2 hold. There exists a unique equilibrium.*

Since the evolutions of η_t and $\{\eta^E_{i,t}, \eta^H_{i,t} : i \in [0,1]\}$ are solved in terms of only their own levels, we have obtained a closed-form solution to the equilibrium. Using this solution, we can obtain a clean analysis of the endogenous aggregate fluctuations arising in equilibrium.

4 Analysis

This equilibrium generates several new results: (1) when diversification is imperfect, the economy features endogenous aggregate volatility, even without aggregate shocks; (2) aggregate volatility does not disappear even if agents have symmetric risk preferences and perfect markets arise to hedge the endogenous risk; (3) when diversification is perfect, the endogenous volatility disappears.

Proposition 4.1. *Suppose $\phi > 0$. Then, $\mathit{Var}_t(d\log K_t) > 0$ and $\mathit{Var}_t(d\eta_t) > 0$.*

In particular, even though there is no exogenous aggregate risk in the economy, aggregate capital and the wealth distribution will feature volatility. This is because of the endogenous risk arising from $dZ^*_{K,t} = \int_0^1 \omega_{i,t}dW_{i,t}di$, $dZ^*_{E,t} = \int_0^1 \eta^E_{i,t}dW_{i,t}di$, and $dZ^*_{H,t} = \int_0^1 \eta^H_{i,t}dW_{i,t}di$. The positive variance of these endogenous "shocks" relies on Lemma 2.3.

The volatilities of these endogenous shocks are analogous to what Gabaix (2011) and Carvalho and Gabaix (2013) have deemed "fundamental volatility," which empirically aggregates the idiosyncratic volatilities of individual firms against their sizes. In this model, correlations between the idiosyncratic shocks of the individual entities (firms, individual wealths) tend to amplify fundamental volatility beyond what iid shocks coupled with granularity would predict. Importantly, empirical methodologies to extract idiosyncratic shocks as regression residuals have nothing to say about the cross-correlations between these residuals. In particular, one cannot rule out arbitrary correlation structures such as the one presented in this paper.

The result of Proposition 4.1 stems from the fact that unit weights $\omega_{i,t}, \eta^E_{i,t}, \eta^H_{i,t} = 1$ is not possible for every i under Assumption 2, when agents are not freely mobile across locations. Indeed, evolution equations (19)-(20) show that $\eta^E_{i,t}$ and $\eta^H_{i,t}$ are subject to idiosyncratic risk, which stems from the fact that moral hazard requires holding non-trivial amounts of skin in the game. The presence of idiosyncratic risk makes net worths fan out over time. Without free mobility to equalize net worths across locations, the net worth distribution remains non-degenerate. Similarly, equation (15) shows that $\omega_{i,t}$ depends on $\eta^E_{i,t}$ and $\eta^H_{i,t}$ and thus cannot be constant over time either By Lemma 2.3, this implies that weighed sums of the idiosyncratic shocks $\{dW_{i,t} : i \in [0,1]\}$ do not "wash out" in the aggregate.

This non-aggregation is a very sharp failure of the law of large numbers. It is more dramatic than the "granularity" hypothesis of Gabaix (2011), in which the distribution of weights needs to be sufficiently fat-tailed to induce aggregate volatility. My results are more similar to Acemoglu et al. (2012), which argues that an input-output network structure may generate aggregate risk through correlated outcomes. In this sense, the correlation structure assumed in $\{dW_{i,t} : i \in [0,1]\}$ may be an outcome of an underlying network between locations i. The difference here is that $\{dW_{i,t} : i \in [0,1]\}$ is still an idiosyncratic shock, even after accounting for the correlation structure, since $\int_0^1 dW_{i,t} di = 0$ almost-surely. Aggregation failure occurs dynamically, because immobility of agents across locations prevents an equalization of location sizes after idiosyncratic shocks hit.

One possible criticism of Proposition 4.1 is that agents in the economy are not allowed to hedge their exposures to the endogenously arising "shocks" $dZ^*_{E,t}$ and $dZ^*_{H,t}$ (recall: agents can effectively trade on $dZ^*_{K,t}$ through the market portfolio of outside equity). Agents have symmetric risk preferences (which include no hedging demands) and the model has no exogenous aggregate fluctuations. Given the reasoning of Di Tella (2017), the presence of frictionless markets for hedging all aggregate risks, including these endogenous risks, should eliminate volatility in the wealth share η_t. In this economy, that logic fails, however.

Corollary 4.2. *Suppose the economy now includes frictionless markets for hedging $dZ^*_{K,t}$, $dZ^*_{E,t}$, and $dZ^*_{H,t}$. These hedging assets are in zero net supply. The statement of Proposition 4.1 still holds.*

The basic intuition for Corollary 4.2 comes from the basic economy outlined above. In particular, notice that the aggregate risk premium γ_t shows up symmetrically in (28)-(29) and will cancel out in the evolution of η_t. Markets for trading claims on $dZ^*_{E,t}$ and $dZ^*_{H,t}$, which will have aggregate risk prices $\pi^*_{E,t}$ and $\pi^*_{H,t}$, will be irrelevant to the drift of η_t. Likewise, the non-zero diffusion of η_t, which comes from imperfect aggregation rather than imperfect hedging, cannot be eliminated. The key is that agents cannot trade away their idiosyncratic risks, and it is these risks which do not aggregate.

Recall the assumption that $\phi > 0$ in Proposition 4.1. It turns out that if observability and contractibility are perfect ($\phi = 0$), then endogenous risk disappears. The following proposition states the result, which implies the converse of Proposition 4.1 also holds. Hence, aggregate volatility emerges if and only if risk-sharing is imperfect.

Proposition 4.3. *Suppose $\phi = 0$. Then, $Var_t(d\eta_t) = 0$. Furthermore, there exists an equilibrium in which $Var_t(d\log K_t) = 0$ as well. Finally, if $\delta > 0$, then all variables are asymptotically constant as $t \to 0$.*

In this sense, technologies that improve contracting (e.g., monitoring, legal frameworks) are a substitute for mobility in reducing economic fluctuations. It is not necessary to have agents move toward areas of low wealth if their portfolios can extend to those locations from a distance.

Given Proposition 4.3, a natural conjecture is that aggregate volatility falls as risk-sharing improves (i.e., ϕ falls). Is this conjecture correct? The crucial thing is $\pi_{E,t} = \frac{\phi\sigma x_t}{\eta_t}$ is increasing in ϕ. With lower risk compensation, the aggregate expert wealth share η_t drifts downwards, see (18). This can actually increase aggregate volatility along the transition path, especially if $\rho_E > \rho_H$. In that case, the negative drift $\rho_H - \rho_E$ becomes a relatively stronger force. This makes it more likely that capital will be misallocated (i.e., $x_t < 1$) at some point, even though experts' need to hold less risk in capital as ϕ decreases.

Proposition 4.4. *Suppose the economy is such that $x_t = 1$. Suppose ϕ unexpectedly decreases. Then, there exists wealth distributions $\{\eta^E_{i,t}, \eta^H_{i,t} : i \in [0,1]\}$ such that $Var_t(d\log K_t)$ and $Var_t(d\eta_t)$ increase in the future.*

5 Conclusion

In this paper, I have shown that aggregate fluctuations can arise purely from a combination of financial frictions and non-iid idiosyncratic shocks. The spatially-correlated shocks are interpreted as the outcome of some unmodeled interactions between locations (e.g., geography or industry). Without financial frictions, the shocks aggregate, highlighting a direct role for financial development in reducing aggregate volatility. However, it is possible to find situations where financial development can actually exacerbate aggregate volatility before ultimately reducing it.

A Proofs

Proof of Lemma 2.1. See Appendix B of Khorrami (2018). □

Proof of Lemma 2.2. See Appendix B of Khorrami (2018). □

Proof of Lemma 2.3. Suppose $\omega_{i,t} = 1$ for almost every i. Then, standard Itô integration results imply $\int_0^1 \omega_{i,t} dW_{i,t} di = \int_0^1 dW_{i,t} di = 0$ almost-surely by Lemma 2.2. This proves necessity.

Now, suppose there is a positive measure subset of $[0,1]$ such that $\omega_{i,t} \neq 1$. Because $\omega_{i,t}$ is measurable and jointly continuous in (i,t), at least one subset of this subset is connected. Consider this connected sub-subset and denote it by $[a,b]$. I claim that $\mathrm{Var}_t[\int_a^b \omega_{i,t} dW_{i,t} di] > 0$. Indeed, the variance-minimizing choice of $\{\omega_{i,t} : i \in [a,b]\}$, subject to $\int_a^b \omega_{i,t} di = \bar{\omega} > 0$, is $\omega_{i,t} = \bar{\omega}$ for all $i \in [a,b]$; see Gârleanu, Panageas and Yu (2015) for the solution to this problem allowing for point masses also at the endpoints. But using the correlation function $v(i,j) := 1 - 6\min(|i-j|, 1-|i-j|)$, we have $\mathrm{Var}_t[\int_a^b \bar{\omega} dW_{i,t} di] = (1-|b-a|)^2 \bar{\omega}^2 > 0$.

Next, consider any strict subset $\mathcal{S}$ of $[0,1] \backslash [a,b]$. I claim $\mathrm{Var}_t[\int_{\mathcal{S} \cup [a,b]} \omega_{i,t} dW_{i,t} di] > 0$. This follows from the fact that, as $\mathcal{S}$ is a strict subset of $[0,1] \backslash [a,b]$, and as $\omega_{i,t} \geq 0$, we must have $\mathrm{corr}_t[\int_{\mathcal{S}} \omega_{i,t} dW_{i,t} di, \int_a^b \omega_{j,t} dW_{j,t} dj] > -1$. But no two random variables with imperfect correlation can be combined to yield a zero-variance portfolio, as the following calculation reveals. Let $m, n > 0$, let $\mathrm{Var}(X) = \sigma_X$ and $\mathrm{Var}(Y) = \sigma_Y$, and let $\mathrm{corr}(X,Y) > -1$. Then, $\mathrm{Var}(mX, nY) > (m\sigma_X - n\sigma_Y)^2 \geq 0$. This proves sufficiency of $\omega_{i,t} = 1$ for almost every i. □

Proof of Lemma 3.1. First, when holding capital, agents earn the cash flow

$$k_{i,t}^g [(a_g - r_t - (1-\phi)\gamma_t) dt + \phi\sigma dW_{i,t}], \quad g \in \{E, H\}.$$

Given log risk preferences, optimal capital choices are then given by the Merton formulas

$$\frac{k_{i,t}^E}{n_{i,t}^E} = \frac{\pi_{E,t}}{\phi\sigma} \quad \text{and} \quad \frac{k_{i,t}^H}{n_{i,t}^H} = \frac{\pi_{H,t}}{\phi\sigma}, \tag{21}$$

where $\pi_{E,t}$ and $\pi_{H,t}$ are defined in (11). The optimality of these portfolios in (21) follows from standard log utility portfolio choice, with shorting constraints, as in Cvitanić and Karatzas (1992).

To obtain x_t, aggregate the optimality conditions of experts:

$$x_t = \eta_t \frac{a_E - r_t - (1-\phi)\gamma_t}{\phi^2 \sigma^2}. \tag{22}$$

By doing the same for households' optimality condition, we obtain

$$1 - x_t = (1-\eta_t) \frac{[a_H - r_t - (1-\phi)\gamma_t]^+}{\phi^2 \sigma^2}.$$

By summing these conditions, we obtain an equation for r_t. To write this equation explicitly, note that $x_t < 1$ if and only if $r_t < a_H - (1-\phi)\gamma_t$, which by using the equation for r_t, occurs if and only if

$$\eta_t < \eta^* := \frac{\phi^2\sigma^2}{a_E - a_H}. \tag{23}$$

Thus,

$$r_t = \mathbf{1}_{\{\eta_t<\eta^*\}}\Big[\eta_t a_E + (1-\eta_t)a_H - \phi^2\sigma^2\Big] + \mathbf{1}_{\{\eta_t\geq\eta^*\}}\Big[a_E - \eta_t^{-1}\phi^2\sigma^2\Big] - (1-\phi)\gamma_t. \tag{24}$$

Substituting equation (24) back into (22), we solve completely for x_t. The result is (14).

Finally, to determine the distribution of capital shares $\{\omega_{i,t} : i \in [0,1]\}$, use (21) and the definitions of $(\eta^E_{i,t}, \eta^H_{i,t})$ to obtain (15). □

Proof of Lemma 3.2. First, note that clearing the goods market (under Definition 1), we obtain

$$[x_t a_E + (1-x_t)a_H]K_t - I_t = [\eta_t\rho_E + (1-\eta_t)\rho_H]K_t, \tag{25}$$

where $I_t := \int_0^1 [k^E_{i,t}\iota^E_{i,t} + k^H_{i,t}\iota^H_{i,t}]di$ is aggregate investment. Indeed, given logarithmic utility, consumptions are given by $c^g_{i,t} = \rho_g n^g_{i,t}$ for $g \in \{E,H\}$. Since $N_{E,t} + N_{H,t} = K_t$, we obtain (25).

Next, aggregate capital evolves as follows:

$$\begin{aligned} dK_t &= \int_0^1 dk_{i,t}di \\ &= I_t dt + \sigma K_t \int_0^1 \omega_{i,t}dW_{i,t}di \\ &= K_t[x_t a_E + (1-x_t)a_H - \eta_t\rho_E - (1-\eta_t)\rho_H]dt + K_t\sigma dZ^*_{K,t}, \end{aligned}$$

where the second line uses the definition of I_t, and the last line uses (25) and the definition $dZ^*_{K,t} := \int_0^1 \omega_{i,t}dW_{i,t}di$.

Then, using equity market clearing (under Definition 1), the market portfolio of outside equity consists of the following weights on location i:

$$\lambda_{i,t} = \omega_{i,t},$$

Thus, the outside equity return (4) aggregates to (5), where $dZ^*_{R,t} := \int_0^1 \lambda_{i,t}dW_{i,t}di = dZ^*_{K,t}$.

Finally, we obtain the risk premium γ_t. For agents' holdings of the market portfolio of equity, we have the optimal Merton portfolio formula

$$\theta^g_{i,t} = \frac{\gamma_t}{\sigma^2 z_t^2}.$$

When $z_t = 0$, the equity portfolio is riskless and $\gamma_t = 0$ (in that case, we must set $\theta^g_{i,t}$ to satisfy equity market clearing). To determine γ_t, we clear the equity market (under Definition 1) using this optimal portfolio and obtain (17). $\square$

Proof of Lemma 3.3. First, we obtain the evolutions of $n^E_{i,t}$ and $n^H_{i,t}$, the net worths at location i. These are the same as individual net worths, after substituting the optimality conditions of experts and households, plus the exogenous mobility from Assumption 2:

$$\begin{aligned} dn^E_{i,t} &= n^E_{i,t}\Big[r_t - \rho_E - \delta + (1-\phi)\gamma_t + \pi^2_{E,t}\Big]dt + \delta N_{E,t}dt \\ &\quad + n^E_{i,t}\Big[(1-\phi)\sigma dZ^*_{K,t} + \pi_{E,t}dZ^*_{E,t} + \pi_{E,t}dW_{i,t}\Big] \end{aligned} \tag{26}$$

$$\begin{aligned} dn^H_{i,t} &= n^H_{i,t}\Big[r_t - \rho_H - \delta + (1-\phi)\gamma_t + \pi^2_{H,t}\Big]dt + \delta N_{H,t}dt \\ &\quad + n^H_{i,t}\Big[(1-\phi)\sigma dZ^*_{K,t} + \pi_{H,t}dZ^*_{H,t} + \pi_{H,t}dW_{i,t}\Big]. \end{aligned} \tag{27}$$

Because $\pi_{E,t}$ and $\pi_{H,t}$ are location-invariant, these aggregate to

$$dN_{E,t} = N_{E,t}\Big[r_t - \rho_E + (1-\phi)\gamma_t + \pi^2_{E,t}\Big]dt + N_{E,t}\Big[(1-\phi)\sigma dZ^*_{K,t} + \pi_{E,t}dZ^*_{E,t}\Big] \tag{28}$$

$$dN_{H,t} = N_{H,t}\Big[r_t - \rho_H + (1-\phi)\gamma_t + \pi^2_{H,t}\Big]dt + N_{H,t}\Big[(1-\phi)\sigma dZ^*_{K,t} + \pi_{H,t}dZ^*_{H,t}\Big], \tag{29}$$

Next, apply Itô's formula to the definition $\eta_t := N_{E,t}/(N_{E,t} + N_{H,t})$. The result is (18), where

$$\begin{aligned} \bar{\Psi}_{E,t} &:= \int_0^1\int_0^1 \Big(\eta^E_{i,t}\pi_{E,t} + \omega_{i,t}(1-\phi)\sigma\Big)\Big(\eta^E_{j,t}\pi_{E,t} + \omega_{j,t}(1-\phi)\sigma\Big)v(i,j)\,dj\,di \\ &= \mathrm{Var}_t(d\log N_{E,t})/dt \\ \bar{\Psi}_{H,t} &:= \int_0^1\int_0^1 \Big(\eta^H_{i,t}\pi_{H,t} + \omega_{i,t}(1-\phi)\sigma\Big)\Big(\eta^H_{j,t}\pi_{H,t} + \omega_{j,t}(1-\phi)\sigma\Big)v(i,j)\,dj\,di \\ &= \mathrm{Var}_t(d\log N_{H,t})/dt \\ \bar{\Psi}_{E,H,t} &:= \int_0^1\int_0^1 \Big(\eta^E_{i,t}\pi_{E,t} + \omega_{i,t}(1-\phi)\sigma\Big)\Big(\eta^H_{j,t}\pi_{H,t} + \omega_{j,t}(1-\phi)\sigma\Big)v(i,j)\,dj\,di \\ &= \mathrm{Cov}_t(d\log N_{E,t}, d\log N_{H,t})/dt \end{aligned}$$

represent the quadratic variations between $d\log N_{E,t}$ and $d\log N_{H,t}$, as claimed. The function $v(i,j) := 1 - 6|i-j|(1-|i-j|)$ is the shock correlation from Assumption 1. Importantly, notice that these objects can be computed using only knowledge of $(\eta_t, \{\eta^E_{i,t}, \eta^H_{i,t} : i \in [0,1]\})$.

Similarly, using Itô's formula on the definitions $\eta^E_{i,t} := n^E_{i,t}/N_{E,t}$ and $\eta^H_{i,t} := n^H_{i,t}/N_{H,t}$, we obtain

(19) and (20), where

$$\begin{aligned}
\Phi_{E,i,t} &:= \int_0^1 \Big(\eta^E_{j,t}\pi_{E,t} + \omega_{j,t}(1-\phi)\sigma\Big)v(i,j)dj = \mathrm{Cov}_t(dW_{i,t}, d\log N_{E,t})/dt \\
\bar{\Phi}_{E,t} &:= \int_0^1 \eta^E_{i,t}\Phi_{E,i,t}di = \mathrm{Cov}_t(dZ^*_{E,t}, d\log N_{E,t})/dt \\
\Phi_{H,i,t} &:= \int_0^1 \Big(\eta^H_{j,t}\pi_{H,t} + \omega_{j,t}(1-\phi)\sigma\Big)v(i,j)dj = \mathrm{Cov}_t(dW_{i,t}, d\log N_{H,t})/dt \\
\bar{\Phi}_{H,t} &:= \int_0^1 \eta^H_{i,t}\Phi_{H,i,t}di = \mathrm{Cov}_t(dZ^*_{H,t}, d\log N_{H,t})/dt
\end{aligned}$$

are quadratic variations arising from the endogenous shocks. As with $d\eta_t$, notice that these objects can be computed using only knowledge of $(\eta_t, \{\eta^E_{i,t}, \eta^H_{i,t} : i \in [0,1]\})$. Consequently, the joint process $(\eta_t, \{\eta^E_{i,t}, \eta^H_{i,t} : i \in [0,1]\})_{t\geq 0}$ is an infinite-dimensional diffusion process. □

Proof of Theorem 3.4. The result follows from the facts that (i) with logarithmic utility, all agents' portfolio and consumption choices are uniquely determined as described in the previous lemmas; and (ii) all state variable evolutions are uniquely determined in terms of their own values and those of the other states, i.e., $(\eta_t, \{\eta^E_{i,t}, \eta^H_{i,t} : i \in [0,1]\})_{t\geq 0}$ is an infinite-dimensional diffusion process. □

Proof of Proposition 4.1. First note that Property (iii) of Assumption 1 implies the fields $\{\omega_{i,t} : t \geq 0, i \in [0,1]\}$, $\{\eta^E_{i,t} : t \geq 0, i \in [0,1]\}$, and $\{\eta^H_{i,t} : t \geq 0, i \in [0,1]\}$ are all continuous in (i,t) almost-surely.

Then, it suffices to show that $\phi > 0$ implies $\eta^E_{i,t} \neq 1$ and $\eta^H_{i,t} \neq 1$ on a positive-measure set, for almost any t. Indeed, $dZ^*_{K,t}$, $dZ^*_{E,t}$, and $dZ^*_{H,t}$ depend on the distributions $\{\omega_{i,t} : i \in [0,1]\}$, $\{\eta^E_{i,t} : i \in [0,1]\}$, and $\{\eta^H_{i,t} : i \in [0,1]\}$, respectively. Furthermore formula (15) links these distributions directly, at each location i. Thus, if $\{\eta^E_{i,t} : i \in [0,1]\}$ and $\{\eta^H_{i,t} : i \in [0,1]\}$ both are different from 1 on a positive-measure set, all three distributions are. Using Lemma 2.3, we have that the variances of $dZ^*_{K,t}$, $dZ^*_{E,t}$, and $dZ^*_{H,t}$ are all positive. Consequently, using (16) and (18), the variances of $d\log K_t$ and $d\log \eta_t$ are positive.

To find the positive-measure subset, consider that each i has a corresponding unique location $z(i)$ such that $\mathrm{Cov}[dW_{i,t}, dW_{z(i),t}] = 0$. Consequently, $B_{i,t} := (W_{i,t} - W_{z(i),t})/\sqrt{2}$ is a standard Brownian motion. Thus, the Itô process $d\log\eta^E_{i,t} - d\log\eta^E_{z(i),t} = \pi_{E,t}[(\Phi_{E,z(i),t} - \Phi_{E,i,t})dt + \sqrt{2}dB_{i,t}]$ has non-zero quadratic variation, hence infinite first-order variation. As a result, the set of times $t \in [0,T]$ on which both of the pair $(\eta^E_{i,t}, \eta^E_{z(i),t})$ can be equal to 1 has Lebesgue-measure zero. By extension, this property holds for all rational location indexes $i \in \mathbb{Q} \cap [0,1]$. In other words,

$$\mathrm{meas}\{t \in [0,T] : \eta^E_{i,t} = \eta^E_{z(i),t} = 1 \text{ for any } i \in \mathbb{Q}\cap[0,1]\} = 0, \quad \text{almost-surely.}$$

Consider the complementary set of times, of measure T. Continuity of $\{\eta^E_{i,t} : i \in [0,1]\}$ implies there exists random functions $m(i)$ and $n(i)$, with $m(i) < n(i)$ almost-surely, such that $\eta^E_{j,t} \neq 1$ for all

$j \in (m(i), n(i))$ and all $i \in \mathbb{Q} \cap [0,1]$. The union

$$\mathcal{E} := \bigcup_{i \in \mathbb{Q} \cap [0,1]} (m(i), n(i))$$

is thus a positive-measure set on almost all times t. An identical argument holds for $\eta^H_{i,t}$. □

Proof of Corollary 4.2. One may re-derive the equilibrium assuming markets exist for trading claims on dZ^*_K, dZ^*_E, and dZ^*_H. Importantly, $d\log\eta^E_{i,t} - d\log\eta^E_{j,t}$ and $d\log\eta^H_{i,t} - d\log\eta^H_{j,t}$ have the same non-zero exposure to $dW_{i,t} - dW_{j,t}$ as in the no-hedging equilibrium. Examining the proof of Proposition 4.1, all the arguments go through as before. □

Proof of Proposition 4.3. Using formula (23), we have $\eta^* = 0$ when $\phi = 0$. Hence, as long as $\eta_t > 0$, one can verify that $\pi_{E,t} = \pi_{H,t} = 0$ and $x_t = 0$ for all t. Using formula (18), we have that $d\eta_t$ is locally deterministic. This proves $\text{Var}_t[d\eta_t] = 0$. Furthermore, $\bar{\Psi}_{E,t} = \bar{\Psi}_{H,t} = \bar{\Psi}_{E,H,t} = \sigma^2 z_t^2$. Since $\rho_E > \rho_H$, we have $d\eta_t = \eta_t(1-\eta_t)(\rho_H - \rho_E)dt < 0$. Thus, $\eta_t \to 0$ as $t \to 0$.

It remains to show that, by Lemma 2.3, there exists an equilibrium in which the capital distribution is equal to unity, i.e., $\omega_{i,t} = 1$ for almost all i. But this holds because (i) the aggregate risk in capital can be hedged by experts and households, and (ii) the idiosyncratic risk held as inside equity is zero by $\phi = 0$. Hence, experts are indifferent between holding capital and investing in riskless bonds. It is weakly optimal for all experts to hold the same capital-net-worth ratio.

Finally, using the fact that when $\pi_{E,t} = \pi_{H,t} = 0$, equations (19) and (20) become

$$d\eta^E_{i,t} = \delta(\eta^E_{i,t})^{-1}(1 - \eta^E_{i,t})dt$$
$$d\eta^H_{i,t} = \delta(\eta^H_{i,t})^{-1}(1 - \eta^H_{i,t})dt.$$

Hence, if $\delta > 0$ as stated in Assumption 2, then $\eta^E_{i,t} \to 1$ and $\eta^H_{i,t} \to 1$ for all i. This shows that all variables are asymptotically constant, as claimed. □

Proof of Proposition 4.4. Under construction. □

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