Arbitrage and Beliefs*

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Abstract

We show that when arbitrage opportunities arise in segmented markets, asset prices suddenly become susceptible to self-fulfilling volatility. For this volatility to come about, a key assumption, in addition to segmented markets, is that some stabilizing force keeps price-dividend ratios stationary, which is a natural property of many macrofinance models. For example, if high valuations lead to higher dividend growth rates, even slightly, self-fulfilling dynamics are possible. The dynamics we uncover predict that one asset boom-bust cycle often begets another cycle in a different asset class or geographic location. Finally, the size of arbitrage profits and degree of limits-to-arbitrage are tightly related to the magnitude of self-fulfilling volatility.

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Many empirical studies have documented that capital markets are, at least somewhat, segmented. Not all capital market participants broadly diversify in all markets. And the marginal investor in some markets trades exclusively in them. For example, there is the well known "home bias" among international holdings (French and Poterba, 1991). Investors mainly hold domestic rather than foreign assets. In mortgage-backed securities markets, Gabaix et al. (2007) find that a stochastic discount factor that is based on MBS-specific risk better explains the price of prepayment risk rather than one based on aggregate wealth. Describing the market for catastrophe insurance, Froot and O'Connell (1999) find that most corporations and households self-retain exposures to catastrophic risk. The implication is that the vast majority of catastrophe risk in the US economy is not adequately shared. Evidence of segmentation has also been found in Treasury markets (Hu et al., 2013) and convertible bond markets (Mitchell et al., 2007).

Meanwhile, market segmentation has been routinely cited, both theoretically and empirically, as a crucial reason why arbitrage opportunities develop. Investors with different preferences or beliefs who trade in separate markets are likely to set different prices for the same asset. Without arbitrageurs who can frictionlessly trade between markets, these price discrepancies persist. Chen and Knez (1995) go so far as to theoretically define market segmentation as the presence of arbitrage between markets. In their model, Gromb and Vayanos (2002) rely on segmented markets to form an arbitrage when investors in separate markets are hit with different demand shocks to hold identical assets. Empirically, Ofek et al. (2004) find violations of put-call parity in US stock options that are consistent with segmentation between equity and options markets. Makarov and Schoar (2020) argue that segmentation of cryptocurrency markets explains recurrent arbitrage opportunities between coins from different countries.

In this paper, we uncover a novel aspect of capital market segmentation. We show that when arbitrage opportunities do arise in segmented markets, asset prices suddenly become vulnerable to self-fulfilling volatility. Our model thus helps explain excess volatility in environments where specialists, rather than broadly diversified investors, set prices or where arbitrage capital is slow moving.

In our model, investors are rational and infinitely-lived inside an endowment economy. There are N locations with distinct local asset markets that are segmented. To make our results transparent, we assume that the endowments are locally deterministic, but all our findings continue to hold with aggregate uncertainty. Relative to existing frameworks, our model adds one twist that connects each asset's cash flow growth rate to its valuation. Specifically, the growth rate of an asset's cash flows is assumed to be positively related to its endogenous valuation (price-dividend ratio). We first explain the purpose of this assumption and then discuss its reasonableness.

Without a growth-valuation link, asset prices are *uniquely* determined by their fundamental values because any other price is associated with a violation of transversality. An asset with a price above its fundamental value features a high price-dividend ratio. Prices must continuously rise without bound to satisfy investors and justify the high price. Long-lived investors understand this instability, which is why a unique fundamental value prevails, and self-fulfilling volatility is not possible.

With a growth-valuation link, asset prices still obey fundamental values (our model is bubble-free), but *many* fundamental values can be sustained in equilibrium. Intuitively, if cash flows grow faster when prices rise, investors who trade a richly priced asset today will tolerate future price declines because high future cash flow growth is enough to satisfy their required returns. This growth-valuation link is a stabilizing force that keeps price-dividend ratios stationary. Transversality is generically satisfied, which opens the door to a multiplicity of fundamental valuations and self-fulfilling price volatility.

While unusual on its surface, a link between cash flow growth rates and valuations is actually a natural property of many macrofinance models. One microfoundation comes from the expansive literature on feedback effects between asset prices and corporate decisions (see the survey in Bond et al., 2012). When managers can learn information from stock or bond prices, they incorporate this data into their capital expenditure decisions. The feedback between prices and investment creates a link between publicly available prices and the cash flows underlying those prices. Another microfoundation stems from the literature on "debt overhang" (e.g., Hennessy, 2004). High prices reduce the debt overhang problem and boost investment, which raises growth rates. Yet another microfoundation is from the endogenous growth literature on "creative destruction" (e.g., Aghion and Howitt, 1992). High prices of incumbent firms discourage new firm entry and shrink the obsolescence rate of current products, which raises the growth rate of existing firms' cash flows. We explicitly analyze these latter two microfoundations in our Internet Appendix.

Beyond these several microfoundations, we also emphasize that the relation between cash flows and prices may be mild. For self-fulfilling volatility to occur in our core analysis, growth rates need only be δx% above average when asset prices are x% above average, where δ represents investors' subjective discount rate. For instance, with $\delta = 0.01$, growth rates need to be 0.1% above average when prices are 10% above average.

Given the discussion so far, readers may think that a growth-valuation link alone permits self-fulfilling volatility that has nothing to do with market segmentation or arbitrage. In fact, the presence of many segmented markets and cross-market arbitrage opportunities are essential for self-fulfilling volatility, a point we now address.

First, consider a single-location economy with cash flow y_t. The price-dividend ratio q_t in this economy cannot feature self-fulfilling volatility, even with an assumed link between q_t and the growth rate of y_t. Why? Suppose q_t were to decline for reasons unrelated to fundamental cash flows or discount rates. Having less wealth after the shock, investors will want to cut their consumption c_t. But in a closed economy, $c_t = y_t$. In other words, there is no mechanism to absorb the desired savings by investors, which must be zero in aggregate.

Similarly, in a multiple-location economy evolving under autarky, each location behaves like its own closed economy. Building off the logic from a single-location economy, self-fulfilling volatility cannot exist if all markets are completely segmented.

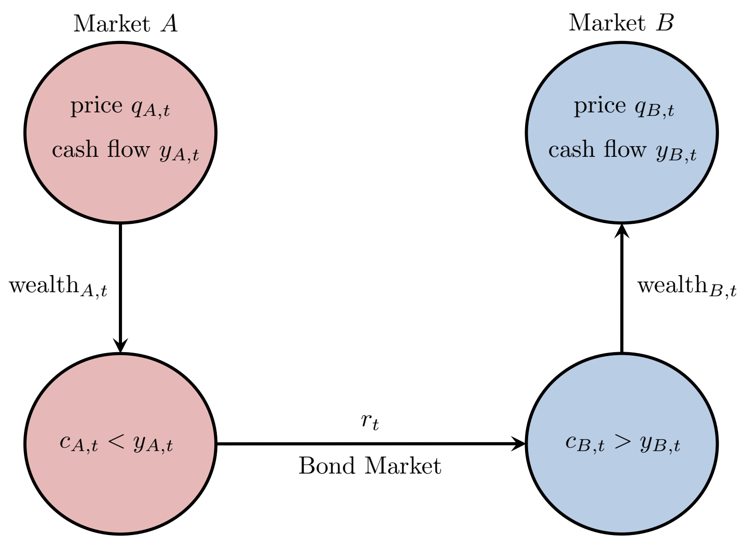
But if there exists an integrated bond market, then self-fulfilling volatility is possible in segmented asset markets. To understand the mechanism, imagine a simplified version of our model with just two markets. One group of investors (A-types) only trades in market A, whereas the second group (B-types) only trades in market B. Suppose the price of asset A declines for reasons unrelated to cash flows or discount rates. Having less wealth after the shock, A-types will want to cut consumption and save a portion of asset A's cash flows in the bond market. By bond market clearing, B-types must borrow this amount and consume more than asset B's cash flows. This consumption plan is only optimal, however, if B-types' wealth has increased, which requires markets A and B to experience equal and opposite changes in value. Thus, the self-fulfilling volatility in our setting is characterized by *redistribution* of wealth between markets—see Figure 1 below.

There are three novel predictions emerging out of this discussion surrounding wealth redistribution. First, asset booms are less likely to be synchronized global phenomena and more likely to be found in individual sectors and geographic locations.

Second, in the two-location example of Figure 1, a self-fulfilling crash in one asset market necessarily precedes a boom in another. More broadly, an asset boom-bust cycle often foreshadows another cycle in a different asset class or different geographic location.

Third, wealth redistribution suggests a deep connection between self-fulfilling volatility and arbitrage. If shocks to assets A and B are offsetting, an investor can construct a riskless portfolio containing both. This portfolio must generate arbitrage profits. The reason why is that A-types demand a risk premium on the self-fulfilling volatility of asset A, and B-types similarly demand a premium on asset B. And so, a portfolio that purchases both assets A and B will earn more than the riskless rate, with the magnitude of the arbitrage tightly related to the magnitude of self-fulfilling volatility. We explore this particular prediction at length in the paper.

Figure 1: Wealth Redistribution Mechanism



Normally, the presence of large arbitrage profits encourages relative-value traders to enter. If an arbitrageur could trade freely across segmented markets, the arbitrage would never appear and neither would self-fulfilling volatility. But if the arbitrageur encountered the kinds of frictions articulated in the literature on the limits of arbitrage (e.g., margin requirements, search frictions, myopic clients), some arbitrage profits would remain. As a result, self-fulfilling volatility will not be fully eliminated. Quantitatively, we show that the magnitude of the frictions limiting arbitrage—as measured by the cross-sectional dispersion in state-price densities—disciplines the magnitude of viable self-fulfilling volatility.

Contribution to the literature. Our paper contributes to two groups of literature: (1) limits to arbitrage in financial markets; and (2) theories of self-fulfilling dynamics. The starting point of the limits-to-arbitrage literature is the plethora of observed arbitrage trades. Examples of such trades include spinoffs (Lamont and Thaler, 2003); "on-the-run/off-the-run" bonds (Krishnamurthy, 2002); put-call parity (Ofek et al., 2004); convertible bonds (Mitchell, Pedersen and Pulvino, 2007); covered interest parity (Du, Tepper and Verdelhan, 2018; Du, Hébert and Huber, 2019); Treasury spot and future-implied repo rates (Fleckenstein and Longstaff, 2018); and cryptocurrencies (Makarov and Schoar, 2020). Many theoretical papers in this area have explored specific microfoundations—such as margin requirements (Gromb and Vayanos, 2002; Gârleanu and Pedersen, 2011), myopic performance-based clients (Shleifer and Vishny, 1997), search frictions (Vayanos and Weill, 2008; Duffie and Strulovici, 2012), or incentive constraints (Biais, Hombert and Weill, 2021)—to explain why arbitrage opportunities persist. These studies also typically pay detailed attention to the behavior of arbitrageurs.

Instead, we focus on fundamental traders, positioning the behavior of arbitrageurs to the background. Dávila, Graves and Parlatore (2021) take a similar approach in remaining agnostic about the specific constraints limiting arbitrage. As a benefit of this approach, we analytically develop all pricing implications more fully—including the new insight of multiplicity. Because fundamental traders in the model are fully rational and forward-looking, the excess volatility we showcase is conceptually distinct from that arising from "noise traders" (e.g., De Long et al., 1990a,b, 1991; Vayanos and Vila, 2021). Nevertheless, our mechanism can be thought of as providing a rational foundation for the volatility from noise trading behavior.

International financial markets are a context in which market segmentation is particularly important. As our model studies multiple locations with their own fundamentals, one can naturally take a global perspective. A recent international finance literature assumes cross-sectional segmentation of local equity or sovereign debt markets (Lustig and Verdelhan, 2019), possibly with some global intermediary participating in all of them (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2017). Our model points to the possibility of self-fulfilling volatility in these settings.

Our construction of self-fulfilling equilibria shares a similar flavor to seminal studies that build sunspot shocks around a stable steady state. Indeed, the stabilizing forces we identify render our deterministic steady state locally stable. We differ from this literature in some of the assumptions we adopt—we require neither overlapping generations (e.g., Azariadis, 1981, Cass and Shell, 1983, and Farmer and Woodford, 1997) nor aggregate increasing returns (e.g., Farmer and Benhabib, 1994) to induce stability. Instead, we present several new examples of stabilizing forces. Our equilibrium construction is also more general in permitting an arbitrary numbers of markets and a broad class of self-fulfilling shocks.

We re-emphasize it is impossible to engender self-fulfilling volatility in a one-location (single-asset) version of our economy, even if dividend growth is linked to prices. This distinguishes our mechanism from several other studies that build multiplicity through collateral constraints or other financing frictions (e.g., Krishnamurthy, 2003; Benhabib and Wang, 2013; Miao and Wang, 2018; Schmitt-Grohé and Uribe, 2021). We also demonstrate the novel connection between the size of arbitrage profits—necessarily a multiple-market concept—and the amount of self-fulfilling volatility in prices.

By focusing on asset prices, our paper engages with Gârleanu and Panageas (2020) and Zentefis (2021). As in those models, our multiplicity arises when there are multiple traded assets and some segmentation between them. In the OLG economy of Gârleanu and Panageas (2020), segmentation arises between physical and human capital because unborn investors have a disproportionate claim to human capital but cannot trade before birth. Multiplicity can arise when physical capital shocks are offset by human capital shocks. The authors interpret this relation as stock market volatility, whereas our equilibrium is better interpreted as self-fulfilling volatility in arbitrage trades. In fact, their economy features no arbitrage at all times, whereas one of our main contributions is tightly connecting arbitrage profits and volatility.

In Zentefis (2021), illiquidity in markets with leverage constraints can generate self-fulfilling price dynamics. Our goal here is to embark on a more general dynamic analysis in a canonical setting, so as to uncover the connection between arbitrage and self-fulfilling fluctuations.

Outline. The remainder of the paper proceeds as follows. Section 1 describes the model in a relatively general way, leaving the stochastic properties of cash flow growth rates unspecified. Section 2 uncovers our results on self-fulfilling volatility, focusing on the role of the growth-valuation link as a stabilizing force ensuring transversality and on the redistributive nature of self-fulfilling shocks. Section 3 analyzes the connection between self-fulfilling volatility and arbitrage. Section 4 discusses existing limits-to-arbitrage models. All have difficulty in generating the precise relation between volatility and arbitrage profits uncovered in our model.

1 Model

Setup. An endowment economy is set in continuous time that is indexed by $t \geq 0$. In our core analysis, endowments are locally deterministic. This assumption makes the results more transparent, as the emergence of self-fulfilling volatility does not rely on the presence of fundamental risk. Nevertheless, we show in Internet Appendix B.1 that our results continue to hold in an economy with aggregate shocks.

In this deterministic economy, the aggregate endowment follows

$$dY_t = Y_t g_t dt, \tag{1}$$

with aggregate growth rate g_t. There are N locations in the economy. Each location can stand for a sector, an industry, a country, or a distinct financial market. The endowment of location n is given by $y_{n,t}$, which follows

$$dy_{n,t} = y_{n,t} g_{n,t} dt, \tag{2}$$

with local growth rate $g_{n,t}$. The aggregate endowment is the sum of all local endowments: $Y_t = \sum_{n=1}^{N} y_{n,t}$. Local growth rates are linked to the aggregate growth rate via the adding up condition: $\sum_{n=1}^{N} y_{n,t} g_{n,t} = Y_t g_t$. For now, we otherwise leave the growth rates unspecified. To economize notation, denote a location's endowment share as $\alpha_{n,t} := y_{n,t}/Y_t$.

Regarding financial markets, each location offers a single asset in positive net supply that is a claim to its local endowment $y_{n,t}$. The equilibrium price of asset n is $q_{n,t} y_{n,t}$, where $q_{n,t}$ is the asset's price-dividend ratio. In addition to these N distinct assets, there is a risk-free bond in zero net supply that offers equilibrium interest rate r_t.

A different representative agent lives in each location. Each agent can invest *only* in his or her local asset market and the short-term bond market that is open to everyone. Hence, local financial markets are segmented, but the bond market is integrated. The bond market allows consumption goods to be traded across locations. Importantly, all agents have rational expectations. Therefore, if self-fulfilling fluctuations transpire, they would be rationally anticipated. Agents have infinite lives, logarithmic utility, and discount rate $\delta > 0$. Mathematically, their preferences are

$$\mathbb{E}_0 \Big[\int_0^\infty e^{-\delta t} \log(c_{n,t}) dt \Big]. \tag{3}$$

Clearing of the goods and bond markets is standard: $\sum_{n=1}^{N} c_{n,t} = Y_t$ and $\sum_{n=1}^{N} q_{n,t} y_{n,t} = Q_t Y_t$, where Q_t is the aggregate price-dividend ratio. Because of market incompleteness due to segmentation, the consumption distribution across locations will be a state variable in equilibrium, so we denote $x_{n,t} := c_{n,t}/Y_t$ as the location-n consumption share.

Extrinsic Shocks. With market clearing established, we next describe asset prices. Because the economy is deterministic, if any price fluctuations are stochastic, they must originate from agents' self-fulfilling beliefs. To allow for this kind of volatility, we conjecture that the price-dividend ratio of each location's asset follows a stochastic process

$$dq_{n,t} = q_{n,t} \Big[\mu_{n,t}^q dt + \sigma_{n,t}^q d\tilde{Z}_{n,t} \Big], \tag{4}$$

where $\tilde{Z}_{n,t}$ is a one-dimensional Brownian motion. The economy has no intrinsic uncertainty. The shock $\tilde{Z}_{n,t}$ is therefore *extrinsic*, and it is the source of self-fulfilling asset price volatility, if any exists. Let $\tilde{Z}_t := \left(\tilde{Z}_{n,t}\right)_{n=1}^{N}$ be a vector of all locations' extrinsic shocks.

Economically, the extrinsic $\tilde{Z}$ shocks arise from sources that we do not explicitly model. Investor sentiment or signals that coordinate beliefs might trigger the self-fulfilling fluctuations, in a manner similar to Benhabib et al. (2015). Heterogeneity in opinions between optimists and pessimists akin to Scheinkman and Xiong (2003) can be another source. Correlated institutional demand shocks as described in Koijen and Yogo (2019) can yet be another driver of the price changes. Our goal is to demonstrate when these kinds of sources can move asset prices in a self-fulfilling manner, even when an economy is deterministic and investors have rational expectations.

We allow the extrinsic shocks in the economy to obey an arbitrary correlation structure. A convenient way to represent this structure uses an N-dimensional basis of uncorrelated Brownian motions $Z_t := (Z_{n,t})_{n=1}^{N}$ and an $N \times N$ matrix of constants M that captures their relation. From these two components, we rewrite the vector of extrinsic shocks as

$$\tilde{Z}_t = MZ_t. \tag{5}$$

The matrix M is normalized so that $\operatorname{diag}[MM'] = (1, \ldots, 1)'$, which preserves $\tilde{Z}_t$ as a collection of Brownian motions. Substituting Eq. (5) into Eq. (4) shows that the self-fulfilling shock to asset n at time t is $\sigma^q_{n,t} M_n dZ_t$, where M_n is the n-th row of M.[1]

The matrix M is a crucial object in the model. To illustrate its structure, we consider the following examples, which we use repeatedly throughout the text.

Example 1 (Uncorrelated shocks)**.** Suppose M is the identity matrix. This structure implies $\tilde{Z}_t = Z_t$, which renders all extrinsic shocks uncorrelated.

Example 2 (Two-by-two redistribution)**.** Suppose $N = 2$ and let

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}. \tag{6}$$

This example has two locations and one source of extrinsic uncertainty. The matrix M puts $\tilde{Z}_{1,t} = -\tilde{Z}_{2,t}$, which implies that the self-fulfilling price changes redistribute wealth between the two assets. As one price falls, the other rises.

[1] Although markets are incomplete in the model, they are dynamically complete. The vector $\tilde{Z}_{n,t} = M_n Z_t$ is generated by N distinct shocks, but it suffices for agent n to only trade $\tilde{Z}_{n,t}$, which is the shock that her local asset loads on. Indeed, if we introduce in each market zero-net-supply Arrow securities spanning Z that are traded only in market n, the equilibrium remains unchanged.

Example 3 (General redistribution)**.** This example is the N-dimensional counterpart to Example 2. Let $\tilde{M}$ be an $N \times N$ non-singular matrix. Suppose

$$M = \tilde{M} - \frac{1}{N}\mathbf{1}'\tilde{M} \otimes \mathbf{1}. \tag{7}$$

In this structure, each element of the matrix $\tilde{M}$ is reduced by the simple average of its columns. This operation makes the column sums of M equal zero. The key consequence of this design is that $\mathbf{1}'d\tilde{Z}_t = \mathbf{1}'MdZ_t = 0$ almost-surely. Any other linear combination of $d\tilde{Z}_t$ does not equal 0. As a result, $\text{rank}(M) = N - 1$. In this example, self-fulfilling price changes redistribute wealth across the N markets.

2 Self-fulfilling volatility

At first glance, readers might be divided on whether non-fundamental volatility is possible in the model we have just laid out. On the one hand, no "arbitrageur" exists to connect market dynamics across locations, so what disciplines local market prices? On the other hand, identical fundamental investors inhabit each location, so shouldn't prices share a common fundamental value? Here, we shed light on this issue, clarifying when non-fundamental price dynamics exist and when they do not.

To develop some understanding of the conditions required for self-fulfilling volatility to arise, let us first consider the market clearing conditions. Investors with log utility consume a fraction δ of their wealth, so the aggregate wealth-consumption (price-dividend) ratio is $Q_t = \delta^{-1}$. Bond market clearing can then be written as

$$\sum_{n=1}^{N} \alpha_{n,t} q_{n,t} = \delta^{-1}. \tag{8}$$

Because the aggregate wealth-consumption ratio is constant, if any extrinsic shocks affect $q_{n,t}$, they must be offset by extrinsic shocks to other assets. Hence, if extrinsic shocks influence prices, these shocks must *redistribute* wealth across markets. From a general equilibrium perspective, such dynamics are sensible, as many market movements are redistributive in the short-run when total capital is held constant.

Wealth redistribution is tied directly to $\text{rank}(M)$. By time-differentiating Eq. (8), we see that the loadings on each of the basis extrinsic shocks dZ_t must be zero:

$$\sum_{n=1}^{N} \alpha_{n,t} q_{n,t} \sigma^{q}_{n,t} M_n = 0. \tag{9}$$

Writing Eq. (9) as a matrix equation gives

$$M' v_t = 0, \tag{10}$$

where $v_t = (\alpha_{1,t} q_{1,t} \sigma^q_{1,t}, \ldots, \alpha_{N,t} q_{N,t} \sigma^q_{N,t})'$ is the column vector of volatilities. If the matrix M were full rank, the unique solution to Eq. (10) would be $v_t \equiv 0$ and there would be no self-fulfilling volatility. However, if M is singular, such that $\text{rank}(M) < N$, a non-zero and time-invariant solution $v_t \equiv v^* \neq 0$ exists. In this case, $\psi_t v^*$ also solves Eq. (10) for any scalar process ψ_t. Therefore, a continuum of candidate volatile equilibria exist, all requiring M to be singular—as in Examples 2 and 3, but not Example 1. The following lemma characterizes the precise notion that shocks must redistribute wealth across asset markets. We discuss the implications of this redistributive characterization in more detail at the end of this section.

Lemma 1. *Suppose the economy features volatility, i.e., $(\sigma^q_{1,t}, \ldots, \sigma^q_{N,t}) \neq 0$. Then, self-fulfilling shocks must be redistributive, in the sense that $\text{rank}(M) < N$.*

Proof. See the discussion directly preceding the lemma. □

Remark 1. As a corollary to Lemma 1, a representative-agent economy ($N = 1$) can never experience self-fulfilling volatility (no non-degenerate matrix M can have $\text{rank}(M) < 1$). Multiple segmented markets are critical.[2]

So far, we have characterized the redistributive nature of self-fulfilling volatility, if it exists. But when does such volatility exist? It turns out that only two conditions are necessary: (i) asset prices are positive and bounded and (ii) all consumers survive in the long run. These two requirements together ensure that there is free disposal of assets (i.e., no negative asset prices) as well as no Ponzi schemes (i.e., transversality holds). Below, we will show how these two requirements translate into constraints on growth rates and asset prices. For now, we summarize this discussion with the following theorem, which provides the sufficient conditions for self-fulfilling volatility.

Theorem 1. *Self-fulfilling volatility is possible as long as the resulting price-dividend ratios $\{(q_{n,t})_{n=1}^N\}_{t \geq 0}$ are bounded, positive processes, and $\lim_{T \to \infty} \mathbb{E}_t[e^{-\delta T} x^{-1}_{n,T}] = 0$, for each n. In this case, let the vector v^* be in the null-space of M'. Given an arbitrary scalar process $\{\psi_t\}_{t \geq 0}$, an equilibrium exists with self-fulfilling volatility $\alpha_{n,t} q_{n,t} \sigma^q_{n,t} = v^*_n \psi_t$ for all n. Finally, all equilibria of the economy are bubble-free.*

[2]This result echoes Loewenstein and Willard (2006), who show that the volatility from noise-traders in De Long et al. (1990a) cannot survive bond market clearing. In our model, each location's agent may hold non-zero positions in bonds, as long as the bond market clears in aggregate.

Proof. See Appendix A.1. □

It is important to emphasize that even with self-fulfilling volatility, there are no bubbles, as prices still equal present values of future dividends. Self-fulfilling volatility in the model is thus consistent with classical no-bubble theorems (e.g., Santos and Woodford, 1997; Loewenstein and Willard, 2000) that give conditions under which bubbles are not possible. The remainder of this section sheds light on which model mechanisms preserve the asset price stationarity required by Theorem 1 and which do not.

Determinacy and instability. The following is a benchmark case in which the equilibrium is unique and non-stochastic.

Proposition 1. *Assume constant local growth rates $g_{n,t} = g$. No equilibrium can have self-fulfilling volatility. Indeed, $(\sigma^q_{1,t}, \ldots, \sigma^q_{N,t}) \equiv 0$ for all t. All assets have identical, constant price-dividend ratios $q_{n,t} = \delta^{-1}$.*

Proof. See Appendix A.2. □

Even though no arbitrageur connects locations, Proposition 1 shows that the presence of rational fundamental traders in each location is enough to pin down asset prices uniquely. In the model, an equilibrium with $q_{n,t} = \delta^{-1}$ always exists, but here, it is also the only one. The reason for this strong determinacy is the instability of price-dividend dynamics when local dividend growth rates are identical and constant. To see this, consider the deterministic model with $(\sigma^q_{1,t}, \ldots, \sigma^q_{N,t}) \equiv 0$. In this case, there is no risk compensation, and all assets must earn the riskless rate. Specifically,

$$\underbrace{\dot{q}_{n,t}/q_{n,t} + g}_{\text{capital gain}} + \underbrace{1/q_{n,t}}_{\frac{\text{dividend}}{\text{price}}} = r_t. \tag{11}$$

Furthermore, since individual consumption paths are deterministic, the interest rate is solely determined by time-discounting and economic growth: $r_t = \delta + g$. Substituting this expression into Eq. (11) gives for each location

$$\dot{q}_{n,t} = -1 + \delta q_{n,t}. \tag{12}$$

This equation represents a dynamical system featuring a single steady state that is unstable. The instability implies that if the price-dividend ratio is below (above) δ^{-1}, it drifts downwards (upwards) at a pace that accelerates over time. This accelerating drift towards positive or negative infinity violates the boundedness of price-dividend ratios in Theorem 1 that is required for self-fulfilling volatility to emerge. Adding shocks (i.e., supposing $\sigma^q_{n,t} \neq 0$) does nothing to remedy the core non-stationarity of the system.

Multiplicity and stability. For any self-fulfilling volatility to exist, Theorem 1 implies that a *stabilizing force* must be present to keep asset prices stationary. We provide a core example of such a force, and discuss alternatives in the appendix. The multitude of examples is meant to demonstrate that the required stability is a natural property of macrofinance models.

In our core example, we assume local growth rates are endogenous and increase with local asset prices. In particular, local growth rates satisfy

$$g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1}), \quad \text{with} \quad \lambda > \delta^2. \tag{13}$$

The exact mathematical connection between growth rates and prices in Eq. (13) is modest. For a standard discount rate of $\delta = 0.01$, local growth rates must be at least 0.1% above average when local valuations are 10% above average.

Eq. (13) is a reduced-form representation of a microfounded, positive link between dividend growth and asset prices. One microfoundation of this link is that prices carry information that affects corporate investment decisions. Examples of this mechanism are in Chen et al. (2007); Bakke and Whited (2010); Goldstein and Yang (2017), where managers learn information from stock market prices and incorporate this information into their investment choices. See also the review in Bond et al. (2012).

Internet Appendix B provides two additional microfoundations of growth-valuation links that work as stabilizing forces. In Internet Appendix B.2, we show that under-investment, of the type induced by "debt overhang" (e.g., Hennessy, 2004; DeMarzo et al., 2012), creates the needed stability. The main idea is that potential gains from investment are high relative to actual investment, which leaves some surplus on the table. As prices rise and boost investment, debt overhang problems shrink, and some of this surplus is captured by local investors. The extra returns gained this way compensate investors for lower dividend yields and ensure stable price-dividend ratios. An intriguing implication is that under-investment can be a self-fulfilling phenomenon for reasons other than those previously identified (e.g., non-convex technologies or borrowing constraints).

In Internet Appendix B.3, we show that an overlapping generations economy with "creative destruction" (e.g., Gârleanu and Panageas, 2020) also produces the required stability. Creative destruction here is represented as new firms entering and displacing incumbents. If the amount of creative destruction is itself a function of asset prices, high asset prices can be self-fulfilled by a small amount of new firm entry, and vice versa. High valuations reduce dividend yields to investors, but living cohorts are compensated with the preservation of their firms, which removes the need for valuations to continue growing and thus creates stability.

Economically, Eq. (13) and the examples in Internet Appendix B share a common property: when prices are high and dividend yields are low, investors are compensated somehow. This compensation can take the form of higher dividend (and, hence, consumption) growth rates, a drop in under-investment wedges, or less creative destruction. It is likely that many other examples of stabilizing forces also exist. By studying several, we stress that a wide range of plausible environments all generate a similar type of stability that can support self-fulfilling volatility when financial markets are segmented.

With this in mind, the next proposition demonstrates the existence of self-fulfilling volatility when local growth rates obey the process in Eq. (13).

Proposition 2. *Assume local growth rates satisfy Eq. (13). Then, self-fulfilling volatility is possible. Specifically, there exists a non-zero process* $\{\psi_t\}_{t\geq 0}$ *such that an equilibrium exists with* $\alpha_{n,t}q_{n,t}\sigma^q_{n,t} = v^*_n\psi_t$ *for all* n*, where* v^* *is in the null-space of* M'.

Furthermore, denote the cross-sectional minimums $\underline{\alpha}_t := \min_n \alpha_{n,t}$, $\underline{x}_t := \min_n x_{n,t}$, *and* $\underline{q}_t := \min_n q_{n,t}$. *The process* ψ_t *can be any bounded process that satisfies the two following conditions:*

(P1) $\psi_t/\underline{\alpha}_t$ *and* $\psi_t/\underline{x}_t$ *are bounded;*

(P2) ψ_t *vanishes as* $\underline{q}_t$ *approaches* $\delta(\epsilon + \lambda^{-1})$ *from above, for some* $0 < \epsilon < \delta^{-2} - \lambda^{-1}$.

Proof. See Appendix A.3. □

The dependence of dividend growth on asset prices allows for self-fulfilling expectations of future price changes to take hold. For instance, if investors anticipate high prices, their expectations for dividend growth rates rise, which support a stable price-dividend ratio and confirms the initial expectations. Conversely, if investors anticipate low prices, expected growth rates drop as well, stabilizing the price-dividend ratio and fulfilling the starting beliefs.

Mathematically, the endogeneity of dividend growth rates acts as the required stabilizing force. To see this clearly, substitute $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$ in place of g in the pricing condition in Eq. (11) and use the fact that $r_t = \delta + g$. The linear price-dividend differential equation of Eq. (12) is then replaced by the quadratic Riccatti equation:

$$\dot{q}_{n,t} = -1 + \delta(1 + \lambda/\delta^2) q_{n,t} - \lambda q_{n,t}^2. \tag{14}$$

From Eq. (14), the dynamical system of the economy now has two steady states. As long as $\lambda > \delta^2$, the larger of the two steady states is the relevant one (i.e., the one with $q_n = \delta^{-1}$). This larger steady state is locally stable, in the sense that $\frac{\partial \dot{q}_n}{\partial q_n}\big|_{q_n=\delta^{-1}} = \delta(1 - \lambda/\delta^2) < 0$. When the economy has such a stabilizing force, some amount of self-fulfilling volatility driven by ψ_t becomes possible. The amount of volatility is only restricted by the requirement that it vanishes when the economy is "far from the steady state" so that the stabilizing force enters unabated. This vanishing property of ψ_t is the essence of properties (P1) and (P2) in Proposition 2.

The role of the bond market. Any self-fulfilling equilibrium of the economy crucially requires an integrated bond market. Indeed, the bond market is the mechanism through which the wealth redistribution of Lemma 1 takes place. Without the bond market (i.e, in autarky), agent n only consumes the cash flows from his or her local endowment $y_{n,t}$. As we will show, since this consumption is locally deterministic, no asset price volatility can be justified. In contrast, if the bond market is open to everyone, agent n can send and receive consumption amounts across locations, with the promise of inter-temporal payback. This availability of trade opens the door for stochastic individual consumption profiles ($dc_{n,t}$ loads on $d\tilde{Z}_{n,t}$), which then creates a stochastic local pricing kernel and justifies price volatility ($dq_{n,t}$ loads on $d\tilde{Z}_{n,t}$).

To see the link between price volatility and the pricing kernel, note that any self-fulfilling volatility must be compensated. Agent n holds exposure to the extrinsic shock $\tilde{Z}_{n,t}$ through her exposure to asset price $q_{n,t}$. Define $\tilde{\pi}_{n,t}$ as the risk price (the Sharpe ratio) associated with this shock, and recall the endowment and consumption shares $\alpha_{n,t} := y_{n,t}/Y_t$ and $x_{n,t} := c_{n,t}/Y_t$. Then, equilibrium in the asset market implies

$$\tilde{\pi}_{n,t} = \delta\Big(\frac{\alpha_{n,t} q_{n,t}}{x_{n,t}}\Big)\sigma_{n,t}^q. \tag{15}$$

Intuitively, $\alpha_{n,t} Y_t q_{n,t} \sigma_{n,t}^q$ is agent n's total exposure to the extrinsic $\tilde{Z}_{n,t}$ shocks, and $\delta^{-1} x_{n,t} Y_t$ is the agent's wealth. For agents with log utility, their required compensation for Brownian shocks is their exposure per unit of wealth. Dividing the exposure by wealth gives the risk price $\tilde{\pi}_{n,t}$. Eq. (15) shows how Sharpe ratios are linked to self-fulfilling volatility: $\sigma_{n,t}^q > 0$ if and only if $\tilde{\pi}_{n,t} > 0$ as well.

Remark 2 ($N = 1$ economy, redux)**.** Recall Remark 1, which stated that self-fulfilling volatility is impossible in a one-location economy. The discussion above provides another intuition for this result. Even if *aggregate* growth rates are endogenous to prices as in Eq. (13)—e.g., aggregate growth is $g_t = G(Q_t)$ for some increasing function G—aggregate consumption growth is still deterministic over a small time-intervals dt. Thus, the representative agent demands no risk premium, and from Eq. (15), there is zero asset volatility. Our findings of multiple equilibria, which require several segmented markets, are thus quite distinct from those derived in one-sector economies that feature asset prices linked to the real economy. For example, collateral constraints induce a link between asset prices and output, and these constraints have often been associated with multiplicity even in a one-location economy (e.g., Kiyotaki and Moore, 1997 or Miao and Wang, 2018). Our results should not be confused with those types of models.

Remark 3 (Autarky)**.** If there were no trade in the riskless bond market, our N-location economy would be in autarky. In that case, each location's consumption dynamics would be locally deterministic (i.e., $c_{n,t} = y_{n,t}$, which has no Brownian shocks). Consequently, similar to Remark 2, agents would demand no risk premium on extrinsic shocks, and so Eq. (15) implies zero asset volatility.

Remark 4 (Small open economy)**.** Although the emergence of self-fulfilling volatility requires an open and active bond market, it does not require bond market clearing. Consider a "small open economy" in which the asset market for claims to the stream $\{y_{n,t}\}_{t \geq 0}$ clears for each n, but the bond market does not. All results are unchanged. Intuitively, since the equilibrium interest rate of the closed economy without extrinsic shocks is constant at $r_t = \delta + g$, it plays no role in providing stability. Mathematically, given any exogenous constant rate r and endogenous local growth rates $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$, the counterpart to the price-dividend ODE of Eq. (14) is

$$\dot{q}_{n,t} = -1 + (r - g + \lambda\delta^{-1})q_{n,t} - \lambda q_{n,t}^2, \tag{16}$$

which has a stable steady state (the larger of the two) if and only if $\lambda > \delta^2(1 + \sqrt{\frac{\delta+g-r}{\delta}})$.

Implications of redistribution. Self-fulfilling volatility operates only with redistributive shocks across multiple locations (Lemma 1). We can use this result to develop several implications, particularly concerning observed boom-bust patterns in asset markets. First, the model explains that self-fulfilling booms often occur less widely and more in isolation. In the model, self-fulfilling asset booms cannot be aggregate global phenomena, as aggregate wealth is fixed at $\delta^{-1}Y_t$. Instead, asset booms must occur in a subset of countries or asset markets, which may be why "bubbles" are often found in a specific region or asset class (Brunnermeier and Schnabel, 2015).

Second, redistributive shocks imply that a self-fulfilling market crash in one country or asset class could beget a boom in an alternative country or asset class. There is some evidence for this type of relation. For example, the 1997 Asian financial crisis coincided with the start of a large boom in the US stock market, primarily in technology stocks. Also, the 2000-02 timing of the US stock market crash matched the run up of the US housing market boom. And finally, the 2006-07 housing market downturn coincided with a boom in commodities markets, mainly in oil, as discussed more formally in Caballero et al. (2008). The authors interpret these facts as a migration of a bubble due to "global imbalances." But the model here demonstrates that such a migration could also take place even without bubbles.

3 Arbitrage profits

Having provided the conditions that permit self-fulfilling volatility, we next connect this volatility to arbitrage profits. Section 3.1 shows that the presence of self-fulfilling volatility and arbitrage opportunities are two sides of the same coin. If one is observed, so too is the other. Section 3.2 demonstrates that arbitrage limits discipline the amount of self-fulfilling price fluctuations that are possible.

3.1 Volatility implies arbitrage and vice versa

The self-fulfilling volatility of Theorem 1 is characterized by wealth redistribution, mathematically captured by the condition $\text{rank}(M) < N$. To further understand what this rank condition implies, consider what would happen if a single trader were allowed to participate in all markets. When $\text{rank}(M) < N$, there is some asset that this trader can replicate using the other $N - 1$ assets. But with self-fulfilling volatility, the price of this asset and its replicating portfolio need not move together. In short, this trader would be faced with an *arbitrage opportunity*.

The next theorem reveals that self-fulfilling volatility emerges if and only if an arbitrage opportunity exists. This equivalence result provides a more intuitive characterization of multiplicity than the rank condition on M. The link between self-fulfilling volatility and arbitrage goes beyond the example stabilizing forces provided in Section 2. It applies to any conceivable example where self-fulfilling volatility exists in economies with segemented financial markets. (In other words, the proof of the following theorem does not use anywhere the condition that $\{(q_{n,t})_{n=1}^N\}_{t\geq 0}$ be bounded positive processes.)

Theorem 2. *Self-fulfilling volatility implies an arbitrage. Conversely, if an arbitrage exists, the equilibrium must feature self-fulfilling volatility.*

Proof. See Appendix A.4. □

First, consider the converse statement of Theorem 2: arbitrage implies self-fulfilling volatility. From the contrapositive, if there were no volatility, then all assets earn the risk-free rate r_t, and so, there is no way to combine them into a portfolio that outperforms the riskless rate. This no-arbitrage, no-volatility equilibrium is the only one that can emerge.

Next, consider the first statement of Theorem 2: self-fulfilling volatility implies the existence of an arbitrage. In the proof of the theorem, we examine a portfolio that puts the amount $\delta\alpha_{n,t}q_{n,t}$ in each asset $n = 1, \ldots, N$. By Eq. (9), the volatility of this portfolio is identically zero. The portfolio represents, in effect, a synthetic risk-free bond. In this case, the condition that $\text{rank}(M) < N$ is critical. Even though all assets have positive self-fulfilling volatility, an investor can manufacture a riskless asset from them. The proof shows mathematically why this synthetic bond earns more than the riskless rate, but the basic intuition comes from the fact that each location's investor demands a risk premium on her local asset. A portfolio that is built as a convex combination of components with risk premia must bear a risk premium itself. In fact, the synthetic bond's excess return over the riskless rate r_t is

$$A_t := \sum_{n=1}^{N} x_{n,t}\tilde{\pi}_{n,t}^2 > 0, \tag{17}$$

where $x_{n,t}$ is agent n's consumption share and $\tilde{\pi}_n$ is the Sharpe ratio of a local asset. The amount A_t can be thought of as a measure of *arbitrage profit* in the model. It is the difference between the return on the synthetic bond created from the arbitrage trade and the return on the actual riskless bond. In addition, the amount of arbitrage profit A_t is exactly the difference between the risk-free rate that prevails without self-fulfilling volatility ($r_t = \delta + g$) and the one with self-fulfilling volatility ($r_t = \delta + g - A_t$). One usually reads this negative term in the interest rate as arising from precautionary savings, but here, the term surfaces from the existence of arbitrage and it equals the amount of arbitrage profits.

The interpretation of the arbitrage trade between the actual risk-free bond and the synthetic bond in practice depends on the context. Some common examples that fit are on-the-run versus off-the-run Treasury bonds; collateralized versus uncollateralized lending (with the arbitrage profit at times captured by the TED spread); and deviations from covered interest parity (CIP). Measures of the last two quantities tend to be minimal for much of the time, but can expand to around 3% during financial crisis periods (Fleckenstein and Longstaff, 2018; Du et al., 2018).

The link between self-fulfilling volatility and the existence of arbitrage opportunities extends to a quantitative relation. The next proposition explains that the magnitude of the arbitrage profit A_t is connected to the amount of self-fulfilling volatility. Indeed, Eq. (15) relates volatility to location-specific risk prices, which constitute A_t in Eq. (17).

Proposition 3. *Let $rank(M) < N$. There exists a non-zero vector v^* in the null-space of M' such that the self-fulfilling volatility ψ_t of Theorem 1 satisfies*

$$\psi_t = \frac{\delta^{-1}\sqrt{A_t}}{\sqrt{\sum_{n=1}^{N} x_{n,t}(\frac{v_n^*}{x_{n,t}})^2}} \leq \frac{\delta^{-1}\sqrt{A_t}}{\mathbf{1}'v^*}, \tag{18}$$

where A_t is the arbitrage profit given in Eq. (17). Consequently, the cross-sectional average return volatility across locations $\sigma_t^* := \sum_{n=1}^{N} \frac{\alpha_{n,t}q_{n,t}}{\sum_{i=1}^{N}\alpha_{i,t}q_{i,t}}\sigma_{n,t}^q$ satisfies

$$\sigma_t^* = \delta\psi_t\mathbf{1}'v^* \leq \sqrt{A_t}. \tag{19}$$

Proof. See Appendix A.5. □

The average return volatility σ_t^* across locations, defined in Proposition 3, is a scale-free summary statistic for the amount of self-fulfilling volatility in the model. The tight link to arbitrage profit, $\sigma_t^* \leq \sqrt{A_t}$, is a bonus. To get a sense of the magnitude of self-fulfilling volatility, consider arbitrage profit that range from $A_t \in [0, 0.03]$, which is consistent with the Treasury evidence of Fleckenstein and Longstaff (2018) and the CIP deviations documented in Du et al. (2018). Then, average return volatilities due purely from self-fulfilling beliefs can range from $\sigma_t^* \in [0, 17.3\%]$, which is a quantitatively significant estimate.

3.2 Cross-market trading limits volatility

Proposition 3 provided a link between the amount of self-fulfilling volatility and the size of arbitrage profits. Given this connection, asset volatility should be curbed by cross-market trading that seeks to capture arbitrage profits. In this section, we make this argument precise by developing a notion of *limits to arbitrage*, and we show how these limits bound the degree of volatility.

Motivated by models such as Gromb and Vayanos (2002) and Gârleanu and Pedersen (2011), we assume that cross-sectional risk prices are linked by some amount of relative-value trading by arbitrageurs. To formalize this notion, it is necessary to examine the location-specific risk prices induced by the basis shocks Z_t. Recall Eq. (5), which presented $\tilde{Z}_t = MZ_t$. If $\tilde{\pi}_{n,t}$ is the risk price of asset n (i.e., the location-n marginal utility response to $d\tilde{Z}_{n,t}$), then

$$\pi_{n,t} := \tilde{\pi}_{n,t} M_n \tag{20}$$

is the marginal utility response to dZ_t, where M_n again is the nth row of the matrix M. Note that $\tilde{\pi}_{n,t}$ is a scalar, whereas $\pi_{n,t}$ is a vector.

We make the following reduced-form assumption about these basis risk prices $\pi_{n,t}$:

$$\|\pi_{j,t} - \pi_{i,t}\| \leq \Pi_t \quad \forall i \neq j. \tag{21}$$

When $\Pi_t > 0$, there are *limits to arbitrage*. This terminology is justified by the well-known equivalence between absence of arbitrage and the existence of a stochastic discount factor that prices all assets. In particular, the case of a perfectly integrated market ($\Pi_t = 0$) will correspond to zero arbitrage profits ($A_t = 0$) and, hence, zero self-fulfilling volatility ($\sigma_t^* = 0$).

In microfounded models, the process for Π_t would be linked to fundamental objects, such as arbitrageur wealth, preferences, constraints, and trading costs. For example, one can think of Π_t as arising from margin constraints and the limited wealth that arbitrageurs can deploy to eliminate risk-price differentials. Faced with these frictions, an arbitrageur would find it worth trading only if risk-price differentials became sufficiently large. Bounds like Ineq. (21) pervade most models of limits to arbitrage. For instance, Proposition 2′ in Appendix B of Gârleanu and Pedersen (2011) explicitly shows how margin constraints lead to a range of viable risk premia. Here, we take Π_t as given and do not model the behavior of these arbitrageurs, choosing instead to describe equilibria based on Ineq. (21), which characterizes the extent of available arbitrage opportunities.[3]

Thus far, we have implicitly assumed $\Pi_t = +\infty$, which is tantamount to assuming that infinite frictions restrict any arbitrage trades between local markets. What happens when there is only partial, but not full, market segmentation? The next proposition details the link between the extent of market segmentation and the amount of self-fulfilling volatility.

[3]We also do not modify the market clearing conditions to account for arbitrageur consumption, which can be justified by the idea that infinite trading would occur if $\|\pi_{j,t} - \pi_{i,t}\| > \Pi_t$ held, but zero arbitrage trading would take place otherwise.

Proposition 4. *Let $0 < \Pi_t < +\infty$ and $rank(M) < N$, There exists a non-zero vector v^* in the null-space of M' such that the self-fulfilling volatility ψ_t of Theorem 1 is bounded by*

$$\psi_t \leq \delta^{-1} L_t^{-1} \Pi_t, \tag{22}$$

where $L_t := \max_{(i,j):i \neq j} \left\| x_{i,t}^{-1} v_i^* M_i - x_{j,t}^{-1} v_j^* M_j \right\|$. The cross-sectional average return volatility across locations $\sigma_t^* = \delta \psi_t \mathbf{1}' v^*$ is bounded by

$$\sigma_t^* \leq \mathbf{1}' v^* L_t^{-1} \Pi_t. \tag{23}$$

Proof. See Appendix A.6. □

Intuitively, with significant limits to arbitrage, large amounts of self-fulfilling volatility can emerge because capital is too slow to correct any such price movements. As limits to arbitrage are relaxed, the amount of self-fulfilling volatility gradually vanish. Propositions 3 and 4 are thus similar in that they connect volatility to a quantitative measure of arbitrage efficacy (arbitrage profits and limits to arbitrage, respectively).

Because of the link between self-fulfilling volatility ψ_t and arbitrage profit A_t, the limits to arbitrage implied by Ineq. (21) puts clear and intuitive bounds on A_t. The link also places bounds on equilibrium risk prices, akin to Hansen and Jagannathan (1991), even though our limits to arbitrage assumption in Ineq. (21) is about *relative* risk prices. The following corollary explains.

Corollary 5. *Under the conditions of Proposition 4, risk prices and arbitrage profit are bounded:*

$$\|\pi_{n,t}\| \leq \frac{v_n^*}{x_{n,t}} L_t^{-1} \Pi_t,$$
$$\sqrt{A_t} \leq \Big(\sum_{n=1}^{N} x_{n,t} (\frac{v_n^*}{x_{n,t}})^2 \Big)^{1/2} L_t^{-1} \Pi_t,$$

where $L_t := \max_{(i,j):i \neq j} \left\| x_{i,t}^{-1} v_i^* M_i - x_{j,t}^{-1} v_j^* M_j \right\|$.

Proof. See Appendix A.7. □

To get a quantitative sense of the volatility bounds, we simulate our model under the endogenous dividend growth rates of Proposition 2; i.e.,

$$g_{n,t} = g + \lambda (q_{n,t} - \delta^{-1}). \tag{24}$$

To satisfy the stability requirement that $\lambda > \delta^2$, we set $\lambda = \delta^2 + 0.01$. We study $N = 10$ locations and set the extrinsic shocks in a similar way as Example 3, where the shocks redistribute wealth across locations:

$$M = \frac{N}{\sqrt{N(N-1)}}\left[I_N - \frac{1}{N}\mathbf{1}\otimes\mathbf{1}'\right] \tag{25}$$

$$= \frac{1}{\sqrt{N(N-1)}}\begin{bmatrix} N-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & N-1 & -1 & \cdots & -1 & -1 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & N-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & N-1 \end{bmatrix}.$$

Note that the columns of M sum to zero and have unit norm. It can easily be verified that $v^* = \mathbf{1}$ is the unique element, up to scale, in the null-space of M. For simplicity, we assume locations start equally sized ($\alpha_{n,0} = 1/N$), and we initialize the simulation with equally-wealthy locations ($x_{n,0} = 1/N$ for all n). We also set $\delta = g = 0.02$.

For the exogenous arbitrage bounds, we set Π_t to a time-invariant value of 0.25. The interpretation is that arbitrageurs are only willing to enter and correct Sharpe ratio differentials greater than 0.25. As will be clear shortly, these limits to arbitrage are quantitatively reasonable.

To simulate $\{x_{n,t} : t \geq 0\}$, first note that the dynamics are given by

$$dx_{n,t} = x_{n,t}(1 - x_{n,t})\left[\tilde{\pi}_{n,t}^2 - \sum_{i\neq n}\frac{x_{i,t}}{1 - x_{n,t}}\tilde{\pi}_{i,t}^2\right]dt + x_{n,t}\tilde{\pi}_{n,t}d\tilde{Z}_{n,t}. \tag{26}$$

These dynamics are derived by applying Itô's formula to the definition $x_{n,t} := c_{n,t}/Y_t$, where the dynamics of $c_{n,t}$ are given in Eq. (30) in the Appendix. Because $\tilde{\pi}_{n,t}$ depends on the self-fulfilling volatility, we assume in the simulation that ψ_t is always at its upper bound, subject to vanishing when needed.[4]

Figure 2 presents the average self-fulfilling return volatility σ_t^* across locations and the associated arbitrage profits A_t from the simulation. The average volatility σ_t^* fluc-

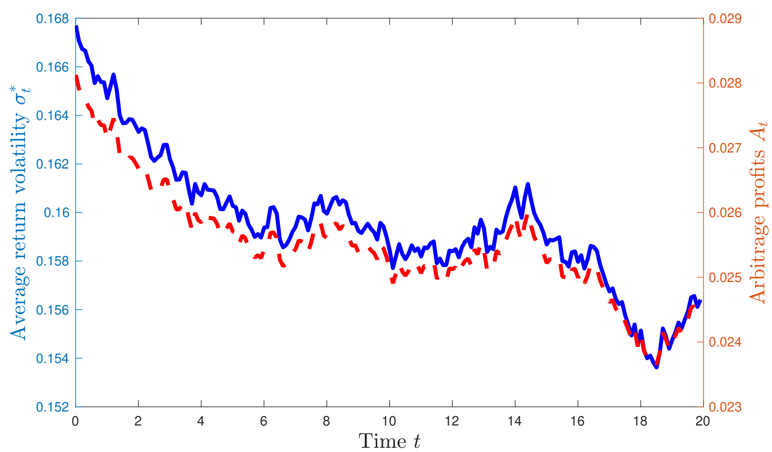
[4]In particular, Proposition 2 shows that ψ_t needs to vanish when $\underline{\alpha}_t$, $\underline{x}_t$, or $\underline{q}_t - \delta\lambda^{-1}$ become low enough. We ensure these conditions by capping the ratio of ψ_t to each of these quantities by 100 in the simulation. In our simulation of $T = 20$ years, none of these vanishing conditions are ever binding, but they would bind in a long-enough simulation. In particular, since $d\alpha_{n,t} = \alpha_{n,t}[g_{n,t} - g]dt$, some locations' endowments can eventually shrink relative to the aggregate (i.e., $\liminf_{T\to\infty}\alpha_{n,T} = 0$ with positive probability). In this case, self-fulfilling volatility must vanish asymptotically. However, this type of long-run degeneracy is not present if the stabilizing force is creative destruction from Internet Appendix B.3.

tuates around 16%. (The volatility could be much lower if we assume the economy is not at the upper bound of the self-fulfilling volatility bound.) Our earlier claim that Π is reasonable is supported by examining the associated simulated arbitrage profits A_t, which are around 2.5%. That value is the upper range of measured arbitrage profits in Fleckenstein and Longstaff (2018). This value can be theoretically verified for the matrix M in Eq. (25) by considering the approximation $x_{n,t} = 1/N$ for all n. Then, the bound simplifies to

$$\sigma_t^* = \sqrt{A_t} \leq \sqrt{\frac{N-1}{2N}}\Pi_t.$$

Substituting $N = 10$ gives $\sigma_t^* \leq 16.8\%$ and $A_t \leq 2.8\%$, which are very close to the ranges displayed in Figure 2.

Figure 2: Simulated Average Return Volatility and Arbitrage Profits



Notes. The figure illustrates simulated values of the average self-fulfilling return volatility σ_t^* across location and the amount of arbitrage profits A_t. Plotted in solid blue against the left axis is the volatility upper bound of Proposition 4 from a simulated economy with $N = 10$ equally-sized locations ($\alpha_n = 1/N$), starting with equal initial wealth ($x_{n,0} = 1/N$), with extrinsic shock matrix M given in Eq. (25) and with endogenous growth rates $g_{n,t}$ from Eq. (24). Plotted in dashed red against the right axis are arbitrage profits A_t from the simulation. The simulation assumes ψ_t is always at the upper bound, except when it needs to vanish, i.e., when $\min_n \alpha_{n,t}$, $\min x_{n,t}$, or $\min_n q_{n,t} - \delta\lambda^{-1}$ become close enough to zero (see Proposition 2). We ensure this takes place by capping the ratio of ψ_t to each of these quantities by 100 in the simulation. Other parameters are described in the text.

4 Discussion of existing models

The model demonstrates a tight link between return volatility and arbitrage profits via self-fulfilling beliefs. But a volatility-arbitrage link might seem consistent with limits to arbitrage in a very general way, going beyond our specific mechanism. To address this possibility, here we discuss some conventional limits-to-arbitrage models and explain why the volatility-arbitrage prediction is not a straightforward implication of those models.

Value-at-risk (VaR) constraints immediately spring to mind as a direct volatility-arbitrage mechanism (e.g., Adrian and Shin, 2010). Higher volatility environments can tighten arbitrageurs' constraints, which limits the amount of trading they can do and makes arbitrage profits higher in equilibrium. Given the relatively high-frequency variation in arbitrage profits seen in the literature (e.g., Du et al., 2018), a VaR-based explanation requires constraints to respond quickly to changes in volatility, which is not necessarily true in practice. For example, whereas the Chicago Mercantile Exchange applies variation margin to profits and losses daily, it adjusts its initial margin requirements relatively infrequently. Given the backward-looking methodologies often used in VaR-based constraints, high-frequency spikes in volatility are unlikely to be fully accounted for in initial margins.[5]

Using fixed margin requirements, Liu and Longstaff (2004) show a positive correlation between the volatility of a relative-value trade—which is modeled as a Brownian bridge that is known to start and end at zero—and the terminal price of a logarithmic agent's optimal portfolio in the trade. Intuitively, high volatility causes large mispricings (i.e., the Brownian bridge is likely to deviate further from zero), which can be exploited by an arbitrageur. And yet, following this sequence of logic carefully, one deduces that high volatility leads to high *future* arbitrage profits, which differs from the contemporaneous relation we prove. Another important omission from their partial equilibrium model is that volatility is not simply an exogenous variable and must come from somewhere in general equilibrium.[6]

A general equilibrium economy that allows arbitrage is the margin-based model of Gârleanu and Pedersen (2011). In their model, holding cash flows fixed, a low-margin claim is more valuable than a high-margin claim. The price discrepancy widens when arbitrageur wealth shrinks, which occurs after negative shocks to the market portfolio (see their Figure 4). However, their model has no noticeable relationship between asset volatilities and arbitrageur wealth.[7] Overall, margin-based asset pricing models

[5]A related alternative, which we cannot rule out, is that arbitrageurs self-impose VaR constraints at either a daily or intra-daily frequency.

[6]See also Kondor (2009), which relates endogenous price gaps to arbitrageur capital.

[7]To see this absence in Gârleanu and Pedersen (2011), one must visually compare their Figures 2 and 3, which plot expected returns and Sharpe ratios as a function of the risk-tolerant agent's consumption share x. Their Figure 2 shows the expected return differential $\mu_{0.4} - \mu_{0.1}$ between an asset with a 0.4 margin requirement and a derivative with a 0.1 margin requirement, with both assets being claims to the same cash flow. Their Figure 3 shows the Sharpe ratio differential $SR_{0.4} - SR_{0.1}$ of these assets. As

do not generally make robust predictions about the relation between arbitrage profits and underlying asset volatilities, primarily because the market-wide shock that shrinks arbitrageur wealth has an ambiguous effect on volatilities.[8]

On the other hand, by allowing a shock directly to non-arbitrageur demands, Gromb and Vayanos (2002) present a general equilibrium margin-based model in which the volatility-arbitrage link could hold, under some assumptions. To understand their mechanism, consider a two-island economy, where the islands' assets share identical cash flows. Local hedgers trade only in their respective islands, whereas an arbitrageur trades in both markets. If one island's local hedgers have a sudden increase in liquidity demands, they sell aggressively in their local market, causing a large discrepancy between their island's asset price and the other island's asset price. The arbitrageur does not fully correct the discrepancy due to a combination of constraints and limited wealth. If the liquidity demand shock is accompanied by an increase in the volatility of future demand shocks, local price volatility would rise as well. A similar equilibrium would emerge if "local hedgers" are replaced in this scenario by "noise traders" (Kyle and Xiong, 2001). This example shows that existing liquidity-based models can, in principle, match the volatility-arbitrage relation we identify, but the explanation requires higher liquidity demand *volatility* at times of large arbitrage profits. In other words, non-arbitrageurs' asset positions must be volatile at these times. We are not aware of any formal empirical evidence for this demand volatility mechanism.

In summary, the existing literature provides some possible mechanisms connecting volatility and arbitrage profits, but the connections are not general, and each comes with caveats. In our view, it is plausible that some piece of the volatility-arbitrage relation that we document is due to self-fulfilling price fluctuations.

an identity, our model has $\mu_{0.4} - \mu_{0.1} = \sigma_{0.4} SR_{0.4} - \sigma_{0.1} SR_{0.1} = \sigma_{0.4}(SR_{0.4} - SR_{0.1}) + (\sigma_{0.4} - \sigma_{0.1}) SR_{0.1}$. Because $\mu_{0.4} - \mu_{0.1}$ and $SR_{0.4} - SR_{0.1}$ share a similar pattern as a function of x, one can conclude that volatilities $\sigma_{0.4}$ and $\sigma_{0.1}$ do not exhibit extreme variation in x. If anything, it appears that $\frac{\mu_{0.4}-\mu_{0.1}}{SR_{0.4}-SR_{0.1}}$ is an inverted U-shaped function of x, consistent with well-known findings of general equilibrium models featuring heterogeneous risk aversions (e.g., Gârleanu and Panageas, 2015).

[8] Gârleanu and Pedersen (2011) do argue (see their footnote 9) that high-margin assets would have higher volatility in equilibrium. So, if the "shock" is to an asset's margin requirement, one would expect an increase in both its volatility and expected excess return (a proxy for arbitrage profits). This link is very similar to VaR constraints, as we discussed earlier. Since margin requirements are somewhat sticky in practice, we are more interested here in the shocks that can lead to higher arbitrage profits, while holding fixed the level of margins.

5 Conclusion

This paper proves in a general setting that self-fulfilling asset price volatility can emerge when financial markets are segmented. The main additional assumption, beyond segmented asset markets, is that a stabilizing force keeps price-dividend ratios stationary. We consider several different examples of such stabilizing forces, and we argue that these forces are common to macrofinance models. These examples are the asset pricing counterparts to the stability conditions provided in seminal papers on sunspots in macroeconomics.

Importantly, the paper demonstrates a strong connection between the availability of arbitrage profits and the possibility of self-fulfilling volatility. Often, the presence of multiple equilibria and self-fulfilling dynamics are viewed as a nuisance for models, but our theoretical result connecting arbitrage and volatility provides foundations for a formal test.

Relatedly, one can test our result that self-fulfilling dynamics redistribute wealth across asset markets. In particular, redistributive dynamics imply that a boom-bust cycle in one asset market will often be followed by another cycle in a different geographic region or asset class, which already has some empirical support.

Given that levels of asset price volatility often far exceed predictions of many theoretical models, the paper's mechanism can help bridge a gap in financial economics. For example, consider corporate equity and bond markets. Although equities and bonds are different claims on the same underlying cash flows, one cannot construct a riskless portfolio between them in a simple way, unlike for covered interest parity, for example. (Under the strong assumption that the underlying shocks affecting both securities and their sensitivities to those shocks are known, one could obtain a no-arbitrage relation between equities and bonds.)

Still, it is entirely possible, and anecdotally true, that equity investors differ from bond investors and that capital is slow moving, whether due to market segmentation or investor habitats. Ma (2019) provides evidence in this direction, suggesting that corporate issuances and buybacks act as a mechanism to profit from price differences. With this in mind, our model suggests that some amount of self-fulfilling volatility is possible in corporate equity and bond markets. In this sense, our focus on risk-free arbitrage is just for clarity, as we can measure the amount of arbitrage profit without having to know investors' pricing kernels. We believe that future research could, through a self-fulfilling mechanism, connect market segmentation to volatility puzzles in other asset markets beyond those featuring risk-free arbitrage.

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Appendix

A Proofs

This appendix contains proofs for the paper and is meant to accompany the text.

A.1 Proof of Theorem 1

To prove the claim, we need to fill in any details that go beyond the discussion following the statement of Theorem 1. There are four brief steps needed to fill in the details.

Step 1: State prices. Each location has its own risk price $\tilde{\pi}_{n,t}$, which is the marginal utility sensitivity to the $d\tilde{Z}_{n,t}$ shock. The state price density for location n is then given by

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t}\Big]. \tag{27}$$

In these terms, we have the no-arbitrage pricing relation:

$$\mu^q_{n,t} + g_{n,t} + \frac{1}{q_{n,t}} - r_t = \sigma^q_{n,t}\tilde{\pi}_{n,t}, \tag{28}$$

which suffices assuming $q_{n,t} > 0$. We can also write these equations in terms of the basis shocks. Let $\pi_{n,t}$ be the risk price vector pertaining to dZ_t, which is potentially location-specific because of market segmentation. The link between these two, by substituting Eq. (5) into Eq. (27), is given in Eq. (20).

Step 2: Optimality. Log utility agents optimally consume δ fraction of their wealth. Investor n's wealth is given by $y_{n,t}q_{n,t} + \beta_{n,t}$, where $\beta_{n,t}$ is her risk-free bond market position. Let $\theta_{n,t} := \frac{y_{n,t}q_{n,t}}{y_{n,t}q_{n,t}+\beta_{n,t}}$ be the fraction of wealth this investor puts in the local risky asset. Note that market clearing is imposed automatically in this formula, as the local investor n holds the entirety of the local asset. Given the dynamic conjecture for asset prices and the consumption-wealth ratio δ, each investor then has consumption dynamics:

$$\frac{dc_{n,t}}{c_{n,t}} = \Big[r_t - \delta + \theta_{n,t}\sigma^q_{n,t}\tilde{\pi}_{n,t}\Big]dt + \theta_{n,t}\sigma^q_{n,t}d\tilde{Z}_{n,t}. \tag{29}$$

Under these assumptions, optimal portfolio choices are given by the standard mean-variance formula $\theta_{n,t}\sigma^q_{n,t} = \tilde{\pi}_{n,t}$. Substituting this portfolio choice into Eq. (29), equilibrium consumption dynamics are

$$\frac{dc_{n,t}}{c_{n,t}} = \Big[r_t - \delta + \tilde{\pi}^2_{n,t}\Big]dt + \tilde{\pi}_{n,t}d\tilde{Z}_{n,t}. \tag{30}$$

From Eq. (27) and Eq. (30), we obtain $\xi_{n,t}c_{n,t} = \xi_{n,0}c_{n,0}\exp(-\delta t)$, so that the static budget constraint (with wealth defined as $w_{n,t} := y_{n,t}q_{n,t} + \beta_{n,t}$)

$$\mathbb{E}_t\Big[\int_0^\infty \frac{\xi_{n,t+s}}{\xi_{n,t}} c_{n,t+s} ds\Big] = w_{n,t} \tag{31}$$

holds automatically with $c_{n,t} = \delta w_{n,t}$. Note that in deriving (31) we have used the individual transversality condition

$$\lim_{T\to\infty} \mathbb{E}_t[\xi_{n,T} w_{n,T}] = 0, \tag{32}$$

as usual.

Step 3: Aggregation. Recall the consumption shares $x_{n,t} := c_{n,t}/Y_t$ and the endowment shares $\alpha_{n,t} := y_{n,t}/Y_t$. Notice that $\theta_{n,t} = \delta\alpha_{n,t}q_{n,t}/x_{n,t}$, which, combined with the optimal portfolio choice, yields equation (15). Time-differentiating the goods market clearing condition $\sum_{n=1}^{N} c_{n,t} = Y_t$ and using (30), we have

$$r_t = \delta + g_t - \sum_{n=1}^{N} x_{n,t}\tilde{\pi}_{n,t}^2 \tag{33}$$

and

$$0 = \sum_{n=1}^{N} x_{n,t}\tilde{\pi}_{n,t}M_n. \tag{34}$$

Substituting (15) into (34) delivers equation (9). Also, combining the asset-pricing equation (28), which is an equation for μ_n^q, with the risk-free rate equation (33), one can show that (8) holds if and only if $\sum_{n=1}^{N} \alpha_n q_{n,0} = \delta^{-1}$, i.e., if an initial restriction holds for prices. In addition, note that consumption share dynamics are obtained by Itô's formula, with the result in equation (26).

Step 4: Free-Disposal, No-Ponzi, Transversality. The fact that $q_{n,t}$ is always positive ensures free-disposal holds. We also require the No-Ponzi condition[9]

$$\lim_{T\to\infty} \xi_{n,T}\beta_{n,T} = 0, \quad \text{a.s.} \tag{35}$$

To prove (35), we will proceed in two steps. First, we first prove a slightly different condition which is a transversality condition on prices (no-bubble condition):

$$\lim_{T\to\infty} \mathbb{E}_t\left[\xi_{n,T} y_{n,T} q_{n,T}\right] = 0. \tag{36}$$

Note that by always working within a class of equilibria that satisfy (36), we are proving that any equilibrium we study must be bubble-free (last statement of the Theorem). Second, we will prove that the transversality condition (36) implies the No-Ponzi condition (35).

[*Proof that (36) holds*]: Recall from the discussion above that $\xi_{n,T} = \xi_{n,0}c_{n,0}e^{-\delta t}/c_{n,T}$, so

$$\xi_{n,T} y_{n,T} q_{n,T} = \xi_{n,0}c_{n,0}e^{-\delta T}\frac{\alpha_{n,T}}{x_{n,T}} q_{n,T}.$$

Note that $\alpha_{n,T}$ is bounded above by 1. Consequently, under the theorem's assumptions—$(q_{n,t})_{t\geq 0}$ is positive and bounded and $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T}x_{n,T}^{-1}] = 0$—condition (36) holds.

[*Proof that (36) implies (35)*]: Since $w_{n,t}$ and $q_{n,t}$ are both positive ($w_{n,t} \geq 0$ by the solvency constraint, and $q_{n,t} \geq 0$ by assumption), and since $\xi_{n,t}$ is the local state-price density, we know $(\xi_{n,t}w_{n,t})_{t\geq 0}$ and $(\xi_{n,t}y_{n,t}q_{n,t})_{t\geq 0}$ are both continuous, positive super-martingales. So by Doob's

[9] Technically, condition (35) should be an inequality "$\geq$" but optimality will impose "$=$" so we immediately write it that way.

super-martingale convergence theorem, we know that $\lim_{T\to\infty}\xi_{n,T}w_{n,T}$ and $\lim_{T\to\infty}\xi_{n,T}y_{n,T}q_{n,T}$ both exist and are finite. Second, conditions (32) and (36) imply there exists a sub-sequence of times $\{T_j\}_{j=1}^{\infty}$ along which $\lim_{j\to\infty}\xi_{n,T_j}w_{n,T_j}=0$ and $\lim_{j\to\infty}\xi_{n,T_j}y_{n,T_j}q_{n,T_j}=0$. But these limits must be the same along any subsequence, by the first step (i.e., that the limits exist), which shows $\lim_{T\to\infty}\xi_{n,T}w_{n,T}=\lim_{T\to\infty}\xi_{n,T}y_{n,T}q_{n,T}=0$. Finally, using the identity $w_{n,T}=y_{n,T}q_{n,T}+\beta_{n,T}$ with these results, we obtain (35).

A.2 Proof of Proposition 1

Given the transversality condition in Eq. (36), we have

$$q_{n,t}=\mathbb{E}_t\Big[\int_t^{\infty}\frac{\xi_{n,s}}{\xi_{n,t}}\frac{y_{n,s}}{y_{n,t}}ds\Big].$$

Using $g_{n,t}=g$ for all (n,t) and $r_t=\delta+g-A_t\leq\delta+g$, where $A_t:=\sum_{n=1}^{N}x_{n,t}\tilde{\pi}_{n,t}^2\geq 0$, we have

$$q_{n,t}=\int_t^{\infty}e^{-\delta(s-t)}\tilde{\mathbb{E}}_t^n\Big[\exp\big(\int_t^s A_u du\big)\Big]ds\geq\int_t^{\infty}e^{-\delta(s-t)}ds=\delta^{-1},$$

where $\tilde{\mathbb{E}}_t^n$ is the location-n risk-neutral expectation, which is mutually absolutely-continuous with respect to $\mathbb{E}$. Using the bond-market clearing condition (8), we must have $q_{n,t}=\delta^{-1}$ for all (n,t).

A.3 Proof of Proposition 2

Consider $g_{n,t}=g+\lambda(q_{n,t}-\delta^{-1})$ with $\lambda>\delta^2$ and fixed ϵ that satisfies $0<\epsilon<\delta^{-2}-\lambda^{-1}$. Supposing $\text{rank}(M)<N$, conjecture a stochastic equilibrium exists with $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q=v_n^*\psi_t$ and $\tilde{\pi}_{n,t}=\delta v_n^*\psi_t/x_{n,t}$ for some process ψ_t. Substituting these and all other equilibrium objects into the asset-pricing equation (28), we have

$$dq_{n,t}=\Big[-1+\Big(\delta+\lambda\delta^{-1}-\delta^2\psi_t^2\sum_{i=1}^{N}\frac{(v_i^*)^2}{x_{i,t}}\Big)q_{n,t}-\lambda q_{n,t}^2+\delta\frac{(v_n^*\psi_t)^2}{\alpha_{n,t}x_{n,t}}\Big]dt+\frac{v_n^*}{\alpha_{n,t}}\psi_t M_n dZ_t. \tag{37}$$

We show that if properties (P1) and (P2) are satisfied, then $q_{n,t}$ remains bounded for all n. As a preliminary, define

$$D(q):=-1+(\delta+\lambda\delta^{-1})q-\lambda q^2. \tag{38}$$

When $\psi_t=0$, all local price-dividend ratios follow $dq_{n,t}=D(q_{n,t})dt$. Note that $D(q)=0$ is a quadratic equation that has two roots: δ^{-1} and $\delta\lambda^{-1}$. Moreover, $D(q)>0$ if and only if $q\in(\delta\lambda^{-1},\delta^{-1})$.

Under property (P2), if $\underline{q}_t=\delta(\epsilon+\lambda^{-1})$, we have $\psi_t=0$ and so

$$d\underline{q}_t=D\big(\delta(\epsilon+\lambda^{-1})\big)dt>0.$$

Note that, under property (P1), the drift and diffusion coefficients of $q_{n,t}$ are bounded, so $q_{n,t}$ is almost-surely path-continuous. This proves that the entire path is bounded below: if $q_{n,0}>\delta(\epsilon+\lambda^{-1})$ for all n, then $\{q_{n,t}\}_{t\geq 0}>\delta(\epsilon+\lambda^{-1})$ for each n almost-surely.

On the other hand, bond market clearing (8), plus this lower bound on valuations, implies an upper bound on the maximal valuation:

$$\bar{q}_t:=\max_n q_{n,t}<\underbrace{\alpha_{\bar{n}_t,t}^{-1}\big[\delta^{-1}-(1-\alpha_{\bar{n}_t,t})\delta(\epsilon+\lambda^{-1})\big]}_{:=b_t},\quad\text{where}\quad\bar{n}_t:=\arg\max_n q_{n,t}.$$

It suffices to show that $\mathbb{P}[\sup_t b_t < +\infty] = 1$. As long as $\underline{\alpha}_t > 0$, we always have $b_t < +\infty$. As a result, we need only consider the case $\underline{\alpha}_t = \alpha_{\bar{n}_t,t}$ (i.e., the location with maximal valuation is the location with minimal endowment share) and suppose $\underline{\alpha}_t \searrow 0$. However, since $\bar{q}_t > \delta^{-1}$, we have

$$d\alpha_{\bar{n}_t,t} = \alpha_{\bar{n}_t,t}\lambda[\bar{q}_t - \delta^{-1}]dt > 0,$$

which contradicts $\underline{\alpha}_t \searrow 0$.

In summary, $\{(q_{n,t})_{n=1}^N : t \geq 0\}$ is positive and bounded almost-surely, so to verify the conditions of Theorem 1, it remains to show that $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} x_{n,T}^{-1}] = 0$. Substituting equilibrium objects into (26), we have

$$dx_{n,t} = \psi_t^2 \delta^2 \Big[(1 - x_{n,t})\frac{(v_n^*)^2}{x_{n,t}} - x_{n,t}\sum_{i\neq n}\frac{(v_i^*)^2}{x_{i,t}}\Big]dt + \psi_t \delta v_n^* M_n dZ_t. \tag{39}$$

Define $\underline{x}_t := \min_n x_{n,t}$ and $\underline{n}_t := \arg\min_n x_{n,t}$. Ignoring local times, $\underline{x}_t$ follows

$$d\underline{x}_t = \psi_t^2 \delta^2 \Big[\frac{(v_{\underline{n}_t}^*)^2}{\underline{x}_t} - O(\underline{x}_t)\Big]dt + \psi_t \delta v_{\underline{n}_t}^* M_{\underline{n}_t} dZ_t.$$

On the set of events $\{\psi_t > 0\}$, we have specified in requirement (P1) that $\psi_t/\underline{x}_t$ be bounded; hence, $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T}\mathbf{1}_{\{\psi_T>0\}}\psi_T \underline{x}_T^{-1}] = 0$, which implies $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T}\mathbf{1}_{\{\psi_T>0\}}\underline{x}_T^{-1}] = 0$. On the complementary event $\{\psi_t = 0\}$, it is clear from the evolution equation that $d\underline{x}_t = 0$. Thus, $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T}\mathbf{1}_{\{\psi_T=0\}}\underline{x}_T^{-1}] = 0$. Putting these pieces together, we verify $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T}\underline{x}_T^{-1}] = 0$.

A.4 Proof of Theorem 2

First, assuming the existence of self-fulfilling volatility, let us find a portfolio that has no risk but pays a positive premium over the riskless rate. Consider a portfolio that goes long $\delta\alpha_{n,t}q_{n,t}$ of each asset $n = 1, \ldots, N$, which costs 1 by equation (8). As stated in equation (28), each asset n has expected excess returns that are given by the product of the location-n risk quantity times the risk price: $\sigma_{n,t}^q \tilde{\pi}_{n,t}$. Using equation (15) to substitute $\tilde{\pi}_{n,t}$, the portfolio excess return is

$$\sum_{n=1}^N x_{n,t}\delta^2\Big(\frac{\alpha_{n,t}q_{n,t}}{x_{n,t}}\Big)^2(\sigma_{n,t}^q)^2 \geq 0,$$

which is strictly positive as long as any self-fulfilling volatility obtains. Using the expression for $\tilde{\pi}_{n,t}$, one can easily verify this expression is equivalent to A_t in (17). At the same time, by equation (9), the portfolio volatility is identically zero. This shows that an arbitrage always emerges if there is self-fulfilling volatility.

Next, the claim that absence of self-fulfilling volatility implies no arbitrage follows from (28), whereby all assets return r_t when $\sigma_{n,t}^q = 0$.

A.5 Proof of Proposition 3

Substituting $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q = \psi_t v_n^*$ from Theorem 1 into location-specific risk prices of (15), and substituting the result into (17), we have

$$A_t = \delta^2\psi_t^2\sum_{n=1}^N x_{n,t}\Big(\frac{v_n^*}{x_{n,t}}\Big)^2$$

By inverting this relationship, the amount of self-fulfilling volatility ψ_t can be inferred from A_t, which gives the equality in (18). The upper bound can be obtained by substituting

$$\sum_{n=1}^{N} x_{n,t}\Big(\frac{v_n^*}{x_{n,t}}\Big)^2 \geq \Big(\sum_{n=1}^{N} x_{n,t}\frac{v_n^*}{x_{n,t}}\Big)^2 = (\mathbf{1}'v^*)^2,$$

which holds by Jensen's inequality. To obtain the equality in (19), substitute (8) into the definition of σ_t^* and use the result from Theorem 1 that $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q = \psi_t v_n^*$. To obtain the inequality, use (18).

A.6 Proof of Proposition 4

Substitute equation (20) into equation (15) to get

$$\pi_{n,t} = \delta\Big(\frac{\alpha_{n,t}q_{n,t}}{x_{n,t}}\Big)\sigma_{n,t}^q M_n.$$

Now, use the result of Theorem 1 that $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q = v_n^*\psi_t$. Combining these equations, we have

$$\pi_{n,t} = \delta v_n^* \psi_t \frac{M_n}{x_{n,t}}. \tag{40}$$

Assumption (21) is equivalent to

$$\delta\psi_t \max_{(i,j):i\neq j}\Big\|\frac{v_i^* M_i}{x_{i,t}} - \frac{v_j^* M_j}{x_{j,t}}\Big\| \leq \Pi_t.$$

Solving for ψ_t, we obtain inequality (22). The bounds for σ_t^* are a direct consequence of (22).

A.7 Proof of Corollary 5

To get the both bounds, begin with the volatility bound (22) of Proposition 4 and use

$$\|\pi_{n,t}\| = \delta\psi_t\frac{v_n^*}{x_{n,t}},$$
$$\sqrt{A_t} = \delta\psi_t\sqrt{\sum_{n=1}^{N} x_{n,t}(\frac{v_n^*}{x_{n,t}})^2}.$$

The expression for $\|\pi_{n,t}\|$ comes from taking the norm of equation (40) and using the fact that MM' has ones on its diagonal (this was a normalization). The expression for $\sqrt{A_t}$ comes from expression (18) in Proposition 3.

Internet Appendix
(Not for publication)

Arbitrage and Beliefs

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B Model extensions

This appendix provides extensions of the main model and is online supplemental material.

B.1 Aggregate shocks

Here, we allow for aggregate shocks hitting the endowments. Location-specific endowments now follow

$$dy_{n,t} = y_{n,t}[g_{n,t}dt + \nu dB_t],$$

where B_t is an aggregate Brownian shock, independent of the extrinsic shocks Z_t (and by extension $\tilde{Z}_t$). We maintain the restriction $\sum_{n=1}^{N} y_{n,t}g_{n,t} = Y_t g_t$. Thus, the aggregate endowment follows

$$dY_t = Y_t[g_t dt + \nu dB_t].$$

Conjecture that local price-dividend ratios follow

$$dq_{n,t} = q_{n,t}\Big[\mu^q_{n,t}dt + \sigma^q_{n,t}d\tilde{Z}_{n,t} + \varsigma^q_{n,t}dB_t\Big],$$

where $(\mu^q_{n,t}, \sigma^q_{n,t}, \varsigma^q_{n,t})$ are determined in equilibrium. We will proceed by making one of two possible assumptions on the tradability of this aggregate shock.

Assumption 1. *One of the following holds:*

(a) there are no additional markets open beyond those assumed so far;

(b) there is an integrated market in which agents frictionlessly trade a zero-net-supply Arrow security that has a unit loading on dB_t.

In both cases of Assumption 1, all previous results on self-fulfilling volatility go through. However, we uncover a surprising nuance: equilibrium is consistent with local assets having nearly arbitrary sensitivities to the aggregate shock.

Proposition 6. *With aggregate shocks, the conclusions of Theorem 1 on $(\sigma^q_{n,t})_{n=1}^N$ continue to hold without modification. Regarding $(\varsigma^q_{n,t})_{n=1}^N$, we have the following. Let $(\phi_{n,t})_{n=1}^{N-1}$ be a collection of arbitrary stochastic processes and set $\phi_{N,t} := -\sum_{n=1}^{N-1}\phi_{n,t}$. Then, there exists an equilibrium with $\alpha_{n,t}q_{n,t}\varsigma^q_{n,t} = \phi_{n,t}$ as long as the resulting $\{(q_{n,t})_{n=1}^N\}_{t\geq 0}$ is a bounded, positive process and $\lim_{T\to\infty}\mathbb{E}_t[e^{-\delta T}x^{-1}_{n,T}] = 0$, for each n. All such equilibria of this economy are bubble-free.*

Before giving a formal proof, we provide the basic sketch of the argument. Because our log agents will still consume δ fraction of their wealth in this environment, equilibrium still satisfies equation (8) such that $\sum_{n=1}^{N} \alpha_{n,t} q_{n,t} = \delta^{-1}$. If we time-differentiate this condition as before, matching diffusion terms leads us to

$$\text{(match } dZ_t \text{ terms)} \quad 0 = \sum_{n=1}^{N} \alpha_{n,t} q_{n,t} \sigma_{n,t}^{q} M_n, \tag{41}$$

$$\text{(match } dB_t \text{ terms)} \quad 0 = \sum_{n=1}^{N} \alpha_{n,t} q_{n,t} \varsigma_{n,t}^{q}. \tag{42}$$

Equation (41) is identical to equation (9), which is why the results of Theorem 1 continue to hold. For equation (42), of course it is possible to have $\varsigma_{n,t}^{q} = 0$ for all n. But we may also set $(\varsigma_{n,t}^{q})_{n=1}^{N-1}$ arbitrarily, so long as $\varsigma_{N,t}^{q}$ offsets these sensitivities. Thus, the volatilities have a similar redistributive flavor as before.

This is indeed an equilibrium, as long as the induced dynamics of price-dividend ratios are stationary. To this end, we can easily extend Propositions 1 and 2 to this setting with aggregate shocks. With common growth rates $g_{n,t} = g$, there will be no multiplicity ($\sigma_{n,t}^{q} = \varsigma_{n,t}^{q} = 0$), as the only prices consistent with the transversality condition are $q_{n,t} = \delta^{-1}$. With growth rates that increase sufficiently quickly in local valuations, we can generate stochastic multiplicity, because all that is required is to have both $\sigma_{n,t}^{q}$ and $\varsigma_{n,t}^{q}$ vanish whenever $\min_n q_{n,t}$ or $\min_n \alpha_{n,t}$ become "too small". We omit the details of these results.[10]

The intuition for self-fulfilling fundamental sensitivities differs depending on whether the shock is hedgable or not. When agents cannot hedge the dB_t shock, the logic is similar to the baseline model: agents adjust their consumption, through the bond market, to their conjecture about how the local asset co-moves with the fundamental shock. When agents trade Arrow securities on dB_t in an integrated market, they do not care whether or not their local asset responds to this shock. Enough hedging and risk-sharing will occur in equilibrium such that individual consumptions all have sensitivity ν to dB_t. Under a particular conjecture about $\varsigma_{n,t}^{q}$, location-n agents will form a hedging plan to undo this exposure. This is self-fulfilling: as long as asset prices move according to the conjecture, the hedging plan was correct.

Proof of Proposition 6. We will nest cases (a) and (b) of Assumption 1 in the following setting. Introduce an Arrow security that pays off $\eta_{n,t} dt + dB_t$ per unit of time, where $(\eta_{n,t})_{n=1}^{N}$ will be determined endogenously. Thus, agent n faces the state-price density process, modified from (27):

$$d\xi_{n,t} = -\xi_{n,t} \Big[r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t} + \eta_{n,t} dB_t \Big]. \tag{43}$$

Let $\theta_{n,t}^{\text{agg}}$ be the fraction of wealth a location-n agent invests in the Arrow security, and let $\theta_{n,t}$ be the fraction of wealth invested in the location-specific capital asset as before. The wealth of agent n has the following dynamics (i.e., the dynamic budget constraint):

$$\frac{dw_{n,t}}{w_{n,t}} = \Big[r_t - \frac{c_{n,t}}{w_{n,t}} + \theta_{n,t} \sigma_{n,t}^{q} \tilde{\pi}_{n,t} + \Big(\theta_{n,t} (\nu + \varsigma_{n,t}^{q}) + \theta_{n,t}^{\text{agg}} \Big) \eta_{n,t} \Big] dt + \theta_{n,t} \sigma_{n,t}^{q} d\tilde{Z}_{n,t} + \Big(\theta_{n,t} (\nu + \varsigma_{n,t}^{q}) + \theta_{n,t}^{\text{agg}} \Big) dB_t. \tag{44}$$

[10] To prove an analogous result to Proposition 2 formally, it is convenient that all locations have equal exposures ν to the aggregate shock, so that $\alpha_{n,t}$ evolves locally deterministically for all n.

To implement (a), where agents are not allowed to trade the Arrow security, we impose a fictitious market clearing condition $\theta^{\text{agg}}_{n,t} = 0$ for all n, which will pin down $\eta_{n,t}$ such that no trading in the Arrow security occurs. From the results of Cvitanić and Karatzas (1992), this implements the same equilibrium as if we never introduced this fictitious market. To implement (b), in which the Arrow market exists and is integrated, we impose $\eta_{n,t} = \eta_t$ for all n and clear the market via $\sum_{n=1}^{N} x_{n,t}\theta^{\text{agg}}_{n,t} = 0$. In both cases, we have the capital market clearing condition $\theta_{n,t} = y_{n,t}q_{n,t}/w_{n,t}$ as before.

Thus, we may nest cases (a) and (b) by solving unconstrained optimization problems for our investors, augmented with the general state-price density process (43) as long as $\eta_{n,t}$ is chosen appropriately. Given the state-price density, the pricing condition (28) is replaced by

$$\mu^q_{n,t} + g_{n,t} + \frac{1}{q_{n,t}} + \nu\varsigma^q_{n,t} - r_t = \sigma^q_{n,t}\tilde{\pi}_{n,t} + (\nu + \varsigma^q_{n,t})\eta_{n,t}, \tag{45}$$

along with the requirement $q_{n,t} > 0$. Because all agents have log utility and effectively solve unconstrained portfolio problems with homogeneous wealth dynamics (44), they all consume δ fraction of their wealth; i.e., $c_{n,t} = \delta w_{n,t}$. Then, as B_t and $\tilde{Z}_{n,t}$ are independent, optimal consumption dynamics (30) are modified to read

$$\frac{dc_{n,t}}{c_{n,t}} = \Big[r_t - \delta + \tilde{\pi}^2_{n,t} + \eta^2_{n,t}\Big]dt + \tilde{\pi}_{n,t}d\tilde{Z}_{n,t} + \eta_{n,t}dB_t.$$

Because $dw_{n,t}/w_{n,t} = dc_{n,t}/c_{n,t}$, we therefore have

$$\tilde{\pi}_{n,t} = \theta_{n,t}\sigma^q_{n,t} = \frac{\delta\alpha_{n,t}q_{n,t}}{x_{n,t}}\sigma^q_{n,t},$$
$$\eta_{n,t} = \theta_{n,t}(\nu + \varsigma^q_{n,t}) + \theta^{\text{agg}}_{n,t} = \frac{\delta\alpha_{n,t}q_{n,t}}{x_{n,t}}(\nu + \varsigma^q_{n,t}) + \theta^{\text{agg}}_{n,t}.$$

The first equation is identical to (15).

Now, we aggregate. First, equation (8) still holds, since agents consume δ fraction of wealth, and since both the bond market and the Arrow markets are in zero net supply. Next, time-differentiate the goods market clearing condition $\sum_{n=1}^{N} c_{n,t} = Y_t$ and match drift and diffusion terms to obtain

$$r_t = \delta + g_t - \sum_{n=1}^{N} x_{n,t}\tilde{\pi}^2_{n,t} - \sum_{n=1}^{N} x_{n,t}\eta^2_{n,t}$$
$$0 = \sum_{n=1}^{N} x_{n,t}\tilde{\pi}_{n,t}M_n$$
$$\nu = \sum_{n=1}^{N} x_{n,t}\eta_{n,t}.$$

Using the expressions for $\tilde{\pi}_{n,t}$ and $\eta_{n,t}$ above, along with the condition $\sum_{n=1}^{N} x_{n,t}\theta^{\text{agg}}_{n,t} = 0$ (which holds in cases (a) and (b) both), we obtain equations (41)-(42). Thus, $\sigma^q_{n,t}$ and $\tilde{\pi}_{n,t}$ are solved exactly as in Theorem 1. Letting $(\phi_{n,t})_{n=1}^{N-1}$ be arbitrary processes, and putting $\phi_{N,t} = -\sum_{n=1}^{N-1}\phi_{n,t}$, we may satisfy (42) by setting $\varsigma^q_{n,t}$ by $\phi_{n,t} = \alpha_{n,t}q_{n,t}\varsigma^q_{n,t}$. As before, this is an equilibrium as long as the transversality condition (36) is satisfied, for which it suffices to show that $q_{n,t}$ is almost-surely bounded and $\lim_{T\to\infty}\mathbb{E}_t[e^{-\delta T}x^{-1}_{n,T}] = 0$ almost-surely. Also as before, transversality (36) holding implies all such equilibria are bubble-free.

It remains to solve for $(\eta_{n,t})_{n=1}^{N}$. In case (a), we use $\theta^{\text{agg}}_{n,t} = 0$ in conjunction with the expression for $\eta_{n,t}$ above to get $\eta_{n,t} = \delta\alpha_{n,t}q_{n,t}(\nu + \varsigma^q_{n,t})/x_{n,t}$. In case (b), we impose $\eta_{n,t} = \eta_t$ for all n, which, after substituting into $\nu = \sum_{n=1}^{N} x_{n,t}\eta_{n,t}$, yields $\eta_t = \nu$. □

B.2 Debt overhang as a "stabilizing force"

In this section, we sketch an economy where firms face an investment problem, subject to neoclassical adjustment costs and debt-overhang. The result is a version of Q-theory, but with underinvestment. Because the predictions of this theory are so well-established, at some points we make reduced-form assumptions to simplify the analysis and illustrate our main points on stability.

Firms. There are a continuum of firms in each location n, each employing a linear technology with productivity a and capital as the sole input. The evolution of firm-level capital is

$$dk_{n,t}^{(j)} = k_{n,t}^{(j)}[\iota_{n,t}^{(j)} - \kappa]dt + k_{n,t}^{(j)}\hat{\sigma}d\hat{B}_{n,t}^{(j)},$$

where ι is the endogenous investment rate, κ is the exogenous depreciation rate, and $B^{(j)}$ is an idiosyncratic Brownian shock. The cost of making investment ιk is given by $\Phi(\iota)k$, where $\Phi(\cdot)$ is a convex adjustment cost function. Thus, the investment-production block has the standard homogeneity property in capital.

For this section only, we denote by $q_{n,t}^{(j)}$ the location-n average value of capital to equity, i.e. "average Q" (this will not be the same as the price-dividend ratio that is called "q" in the main text, because the dividend is output minus investment). Thus, the value of firm j is given by $q_{n,t}^{(j)}k_{n,t}^{(j)}$.

We also assume that all firms have long-term debt outstanding, in fact a perpetuity with a fixed and continuously-paid coupon as in Leland (1994) and its descendent papers, without microfounding the reasons for why (e.g., debt tax shield), as this is unimportant. Furthermore, to keep things simple, we assume existing firms can never issue new debt. Finally, firms default optimally, subject to some default costs that are proportional to the firm's capital (these can be redistributed to households to create no deadweight loss). Under these conditions, a typical finding is (see for example Hennessy, 2004, Proposition 2)

$$\tilde{q}_{n,t}^{(j)} := \text{marginal value of capital to equity} < \text{average value of capital to equity} = q_{n,t}^{(j)}.$$

Moreover, essentially by definition of $\tilde{q}$, the optimal investment satisfies $\tilde{q}_{n,t}^{(j)} = \Phi'(\iota_{n,t}^{(j)})$ (see for example Hennessy, 2004, equation 11). Thus, we see that $q_{n,t}^{(j)} > \Phi'(\iota_{n,t}^{(j)})$. The lack of equality here measures the deviation from neoclassical Q-theory.

Despite this deviation, we have the following property. Since $q_{n,t}^{(j)}$ increases with $\tilde{q}_{n,t}^{(j)} = \Phi'(\iota_{n,t}^{(j)})$, and since Φ is a convex function, we have $\iota_{n,t}^{(j)}$ increasing in $q_{n,t}^{(j)}$. We will furthermore make the reduced-form assumption that $\iota_{n,t}^{(j)} = \iota(q_{n,t}^{(j)})$ for some univariate increasing function $\iota(\cdot)$. This assumption is quite benign as it is typically satisfied in applications, because $\tilde{q}_{n,t}^{(j)}$, hence $q_{n,t}^{(j)}$, will typically be monotonic functions of the underlying firm-level state (e.g., leverage ratio).

In summary, we have the following two firm-level properties under debt overhang:

$$q_{n,t}^{(j)} > \Phi'(\iota_{n,t}^{(j)}) \tag{46}$$

$$\iota'(q_{n,t}^{(j)}) > 0. \tag{47}$$

Condition (46) captures the specific debt-overhang mechanism, whereas condition (47) is much more general and applies in almost any investment model. With a more general contractual structure, DeMarzo et al. (2012) also obtains these two results.

Aggregation. We will now make two assumptions that are mainly for tractability in aggregation. First, when a firm defaults and exits, it is replaced by another firm with the same identity j that inherits the defaulting capital stock. We assume this new entrant issues new debt is such that the aggregate location-n value of debt outstanding is always a constant fraction of total location-n capital; i.e., total location-n value of debt is always $\beta k_{n,t}$. Alternatively, this proportionality of aggregate debt to capital could be ensured by augmenting the model with a time-varying exogenous exit rate, but allowing new entrants to issue debt in an optimal way. Either way, this set of assumptions implies it suffices to study equity.

Second, we make assumptions to avoid studying the full cross-sectional distribution of firms within a location. We assume that properties (46)-(47) also hold in the aggregate at each location, and we will presume a certain approximate aggregation on investment and investment costs. In particular, let us define the appropriate aggregates, for capital, average Q, and investment:

$$k_{n,t} := \int k_{n,t}^{(j)} dj$$
$$q_{n,t} := \frac{1}{k_{n,t}} \int q_{n,t}^{(j)} k_{n,t}^{(j)} dj$$
$$\iota_{n,t} := \frac{1}{k_{n,t}} \int \iota(q_{n,t}^{(j)}) k_{n,t}^{(j)} dj.$$

As an approximation, we assume the existence of functions $(\bar{\iota}, \bar{\Phi})$ such that the following hold:

$$\bar{\iota}(q_{n,t}) \approx \int k_{n,t}^{(j)} \iota(q_{n,t}^{(j)}) dj \tag{48}$$
$$k_{n,t} \bar{\Phi}(\bar{\iota}(q_{n,t})) \approx \int k_{n,t}^{(j)} \bar{\Phi}(\bar{\iota}(q_{n,t}^{(j)})) dj. \tag{49}$$

The nature of these approximations is to say that aggregate location-n investment is solely a function of aggregate average Q, rather than the full cross-sectional distribution of average Q's. Furthermore, we assume the following aggregate versions of properties (46)-(47), i.e.,

$$q_{n,t} > \bar{\Phi}'(\bar{\iota}_{n,t}) \tag{50}$$
$$\bar{\iota}'(q_{n,t}) > 0. \tag{51}$$

We conjecture these properties would go through in a full analysis of equilibrium using the cross-sectional distribution of firm size and Q, but this is beyond the scope of this paper. As we make these aggregation approximations, note that we also assume the functions $(\bar{\iota}, \bar{\Phi})$ are independent of location n.

Stability. Now, we can proceed to study stability. The aggregate portfolio of location-n firms' liabilities (debt plus equity) has value $(\beta + q_{n,t})k_{n,t}$, which is a claim to the profits $\int (a - \Phi(\iota_{n,t}^{(j)}))k_{n,t}^{(j)} dj$. Based on approximation (49), this aggregate profit can be approximately written $(a - \bar{\Phi}(\bar{\iota}(q_{n,t})))k_{n,t}$. Furthermore, the return on this portfolio is deterministic, given that all fundamental shocks are idiosyncratic (hence defaults will be idiosyncratic), and thus the return must equal the riskless bond return r_t in equilibrium. Thus, $q_{n,t}$ evolves deterministically, and the (approximate) valuation equation states

$$\frac{a - \bar{\Phi}(\bar{\iota}(q_{n,t}))}{q_{n,t}} + \bar{\iota}(q_{n,t}) - \kappa + \frac{\dot{q}_{n,t}}{q_{n,t}} = r_t. \tag{52}$$

Lemma 2. *Suppose the number of locations N is large enough, that approximations (48)-(49) hold, and that properties (50)-(51) hold with sufficient gaps between the left- and right-hand-sides (i.e., under-investment is large enough). Then, the equilibrium of the model with debt overhang is locally-stable.*

Proof of Lemma 2. We start with approximate valuation equation (52). Time-differentiate $\dot{q}_{n,t}$ with respect to $q_{n,t}$ and $q_{-n,t}$ to obtain

$$\frac{d\dot{q}_{n,t}}{dq_{n,t}} = r_t + \kappa - \bar{\iota}(q_{n,t}) + \bar{\Phi}'(\bar{\iota}(q_{n,t}))\bar{\iota}'(q_{n,t}) - q_{n,t}\bar{\iota}'(q_{n,t}) + q_{n,t}\frac{dr_t}{dq_{n,t}}$$
$$\frac{d\dot{q}_{n,t}}{dq_{-n,t}} = q_{n,t}\frac{dr_t}{dq_{-n,t}}.$$

We will study these equations in the limit $N \to \infty$, which suffices, because the lemma allows us to later make N large enough.

As $N \to \infty$, one can show that

$$r_t \to \delta - \kappa + \lim_{N\to\infty} \sum_{n=1}^{N} \frac{k_{n,t}}{\sum_{i=1}^{N} k_{i,t}} \bar{\iota}(q_{n,t}),$$

which has zero derivative with respect to $q_{i,t}$ for any i. Substituting this result for r_t, we obtain $d\dot{q}_{n,t}/dq_{-n,t} = 0$ and

$$\frac{d\dot{q}_{n,t}}{dq_{n,t}} = \delta + \underbrace{\lim_{N\to\infty} \sum_{m=1}^{N} \frac{k_{m,t}}{\sum_{i=1}^{N} k_{i,t}} \bar{\iota}(q_{m,t}) - \bar{\iota}(q_{n,t})}_{=0 \text{ in steady state}} - \left[q_{n,t} - \bar{\Phi}'(\bar{\iota}(q_{n,t}))\right]\bar{\iota}'(q_{n,t}).$$

The fact that the middle terms net out to zero in steady state is a consequence of the fact that $dk_{n,t} = k_{n,t}[\bar{\iota}(q_{n,t}) - \kappa]dt$, and all locations must experience the same growth rate $\bar{\iota}(q_{n,t}) - \kappa$ in steady state. Thus, we will have $d\dot{q}_{n,t}/dq_{n,t} < 0$, hence local stability by $d\dot{q}_{n,t}/dq_{-n,t} = 0$, if and only if

$$\left[q_{n,t} - \bar{\Phi}'(\bar{\iota}(q_{n,t}))\right]\bar{\iota}'(q_{n,t}) > \delta.$$

This will be true if properties (50)-(51) hold with sufficient gaps, as assumed. □

B.3 Creative destruction as a "stabilizing force"

In this section, we consider another model that allows multiplicity. We show how an overlapping generations (OLG) "perpetual youth" economy – built upon Blanchard (1985) – augmented with a particular type of creative destruction – similar to Gârleanu and Panageas (2020) – creates a stabilizing force upon which extrinsic shocks can be layered. In particular, if new firm creation is more intense when asset valuations are low, the economy possesses a natural stabilizing force. A possible rationale for this feature is that when capital asset valuations are low, they make labor look relatively attractive, which offers a robust outside option for those new entrepreneurs willing to enter. The contribution relative to Gârleanu and Panageas (2020) is to show how this is possible with an arbitrary number of assets (corresponding to the N locations) whose markets are, in addition, not integrated.

Cohorts, Endowments, Markets. In this model, all agents face a constant hazard rate of death $\beta > 0$, with all dying agents replaced by newborns (in the same location), so that population size is constant at 1. To keep matters simple, assume all locations have identical constant endowment growth rates and no shocks. That said, the endowment growth of an individual agent differs from the aggregate growth rate; this is the crucial ingredient in this model.

In particular, we assume some amount of *creative destruction*. The endowments of living agents decay at rate $\kappa_{n,t}$ (obsolescence rate), while newborn agents arrive to the economy with new trees of total size $\kappa_{n,t} + g$ (or, in per capita units, their individual trees are $(\kappa_{n,t} + g)/\beta$ in size). Specifically, the time-t endowment accruing to location-n agents born at time $s \leq t$ is

$$y_{n,t}^{(s)} = y_{n,t}(\kappa_{n,s} + g) \exp\Big[- \int_s^t (\kappa_{n,u} + g) du \Big].$$

Note that the aggregate endowment follows

$$dy_{n,t} = d\Big(\int_{-\infty}^t y_{n,t}^{(s)} ds \Big) = y_{n,t}^{(t)} dt + \int_{-\infty}^t dy_{n,t}^{(s)} ds = \underbrace{y_{n,t}(\kappa_{n,t} + g)dt}_{\text{newborn entry}} - \underbrace{y_{n,t}\kappa_{n,t}dt}_{\text{obsolescence}} = y_{n,t} g dt.$$

For now, we leave $\kappa_{n,t}$ unspecified, but note that its formulation will be the determinant of whether multiplicity is possible or not.

Agents can only trade in financial markets while alive. In addition to the tradability of claims to local endowments, agents can access a market for annuities that insures their death hazard and provides a stream of $\beta w_{n,t}^{(s)}$ of income per unit of time, where $w_{n,t}^{(s)}$ is the wealth of a location-n agent born at time $s \leq t$. This assumption is standard in perpetual youth models.

Solution. Under these assumptions, one can show that agents consume $\delta + \beta$ fraction of their wealth, so that the bond market clearing condition (8) is replaced by

$$\sum_{n=1}^N \alpha_n q_{n,t} = (\delta + \beta)^{-1},$$

where $q_{n,t}$ is the (aggregated across cohorts) location-n valuation ratio. Let $\xi_{n,t}$ denote the location-n state-price density, which follows

$$d\xi_{n,t} = -\xi_{n,t} \Big[r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t} \Big].$$

We will continue to examine a bubble-free equilibrium, so that

$$q_{n,t} = \mathbb{E}_t \Big[\int_t^\infty \frac{\xi_{n,\tau}}{\xi_{n,t}} \frac{y_{n,\tau}^{(s)}}{y_{n,t}^{(s)}} d\tau \Big] \quad \text{(for any birth-date } s \leq t\text{, this yields the same answer).}$$

Critically, this valuation does not incorporate wealth gains due to entry of future newborns (i.e., this is the value of alive firms). The dynamic counterpart of this valuation equation is, for some diffusion coefficient $\sigma_{n,t}^q$,

$$\frac{dq_{n,t}}{q_{n,t}} = \Big[r_t + \kappa_{n,t} - \frac{1}{q_{n,t}} + \sigma_{n,t}^q \tilde{\pi}_{n,t} \Big] dt + \sigma_{n,t}^q d\tilde{Z}_{n,t}. \tag{53}$$

The equilibrium riskless rate is obtained as follows. The goods market is integrated across locations, so the market clearing condition is given by

$$Y_t = \sum_{n=1}^N y_{n,t} = \sum_{n=1}^N \int_{-\infty}^t \beta e^{-\beta(t-s)} c_{n,t}^{(s)} ds.$$

Optimal consumption dynamics for alive agents are

$$\frac{dc_{n,t}^{(s)}}{c_{n,t}^{(s)}} = \Big[r_t - \delta + \tilde{\pi}_{n,t}^2 \Big] dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t},$$

whereas newborn agents consume

$$\beta c_{n,t}^{(t)} = \underbrace{(\delta+\beta)}_{\substack{\text{cons-wealth}\\ \text{ratio}}} \times \underbrace{(\kappa_{n,t}+g)y_{n,t}q_{n,t}}_{\text{newborn wealth}}.$$

Time-differentiating goods market clearing, and using these results, we obtain

$$r_t = \delta + \beta - \sum_{n=1}^{N} x_{n,t}\tilde{\pi}_{n,t}^2 - (\delta+\beta)\sum_{n=1}^{N} \alpha_n q_{n,t}\kappa_{n,t}. \tag{54}$$

Stability. To see how the stabilizing force works, it is instructive to once again study the deterministic equilibrium in which extrinsic shocks have no volatility. Substituting (54) into (53) with $\sigma_{n,t}^q = 0$, we obtain

$$\dot{q}_{n,t} = \underbrace{-1 + (\delta+\beta)q_{n,t}}_{\text{unstable component}} - \underbrace{\Big[(\delta+\beta)\sum_{i=1}^{N}\alpha_i q_{i,t}\kappa_{i,t} - \kappa_{n,t}\Big]q_{n,t}}_{\text{stabilizing force}} \quad \text{when} \quad \sigma_{i,t}^q = 0 \quad \forall i. \tag{55}$$

The first piece is the unstable component, propelling valuations further and further away from the "steady state" value $(\delta+\beta)^{-1}$. The second piece—capturing the relative amount of creative destruction in location n—is the stabilizing force.

Based on equation (55), we claim that if $\kappa_{n,t}$ decreases sufficiently rapidly as $q_{n,t}$ increases, then valuation dynamics are stable. Let $\kappa_{n,t} = \kappa(q_{n,t})$ for a decreasing function $\kappa(\cdot)$. Denote the steady-state mean and sensitivity of this function by $\bar{\kappa} := \kappa((\delta+\beta)^{-1})$ and $\lambda := -\kappa'((\delta+\beta)^{-1})$, respectively. Then, compute

$$\frac{\partial \dot{q}_n}{\partial q_m}\Big|_{q_i=(\delta+\beta)^{-1}\,\forall i} = \begin{cases} \delta+\beta-\lambda(\delta+\beta)^{-1}(1-\alpha_n)-\alpha_n\bar{\kappa}, & \text{if } m=n; \\ \lambda(\delta+\beta)^{-1}\alpha_m - \alpha_m\bar{\kappa}, & \text{if } m\neq n. \end{cases}$$

Construct the steady-state Jacobian matrix as

$$J := \Big[\frac{\partial \dot{q}_n}{\partial q_m}\Big|_{q_i=(\delta+\beta)^{-1}\,\forall i}\Big]_{1\le n,m\le N}. \tag{56}$$

Local stability of the steady-state can be determined by the eigenvalues of J. By the Gershgorin circle theorem, all of these eigenvalues will have strictly negative real parts if J has negative diagonal elements and is diagonally dominant. This is easily guaranteed by making $\bar{\kappa}$ and λ large enough, meaning the amount of creative destruction and its sensitivity to prices are both large enough. The result is summarized in the following lemma, with the proof omitted.

Lemma 3. *Assume $\bar{\kappa} > \delta+\beta$ and $\lambda > (\delta+\beta)\bar{\kappa}$. Then, all eigenvalues of J have strictly negative real parts. Consequently, the equilibrium of the creative destruction model is locally stable.*