Segmentation and Beliefs: A Theory of Self-Fulfilling Idiosyncratic Risk*

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Abstract

We study a multi-location general equilibrium model with financial market segmentation that permits self-fulfilling fluctuations. In a precise sense, such fluctuations are most often redistributive, but their volatility varies systematically with an aggregate latent factor. We thus provide a coordination-based microfoundation for time-varying idiosyncratic risk. A key assumption of our analysis is that cash flow growth rates (e.g., firm profit growth, asset dividend growth, or country output growth) rise with valuations. We consider two applications: (i) firm dynamics and their risk factor structure; and (ii) exchange rate disconnect in international macroeconomics.

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This paper presents a theory of self-fulfilling volatility. Broadly speaking, we shed light on the following questions. Why are asset prices so volatile, in excess of cash flows and other "fundamentals"? Excess volatility puzzles have received attention in many contexts, still without definitive answers. More specific to our particular framework, what is the source of idiosyncratic uncertainty? And why does idiosyncratic uncertainty vary over time systematically?

We explore a general equilibrium model with two key features—the presence of multiple markets along with a feedback effect between financial markets and the real economy. Our model has N abstract "locations" each of which receives its own endowment. Depending on the application, think of locations as firms, industries, or countries. Each location has an equity market, which trades claims on its local endowment stream. It is this equity valuation that is subject to multiple self-confirming equilibria. The multiplicity comes about because of an assumption connecting fundamentals to prices: the growth rate of a location's endowment is assumed to be positively related to its endogenous valuation (*growth-valuation link*).

The contributions of the paper are (i) to characterize conditions under which multiple equilibria emerge; (ii) to establish common properties of these equilibria; and (iii) to argue that these properties can speak to various empirical patterns.

Growth-valuation link. A key assumption in our analysis is some dependence of growth on asset valuation ratios (from now on, we will simply refer to such ratios as "valuations"). A sufficiently strong dependence allows for self-fulfilling expectations of future price changes to take hold. For instance, if investors anticipate high prices, their expectations for cash flow growth rates rise, which justifies the high prices and confirms the initial expectations. Conversely, if investors anticipate low prices, expected growth rates drop as well, fulfilling the starting beliefs about low prices.

How should one understand our critical growth-valuation link? Our baseline interpretation comes from the expansive literature on feedback effects between asset prices and corporate decisions (see the survey in Bond et al., 2012). When managers can learn information from stock or bond prices, they incorporate this data into their capital expenditure decisions. The feedback between prices and investment creates a link between publicly available prices and the cash flows underlying those prices. This is just one interpretation for our reduced-form growth assumption. As we discuss in the paper, all we need is some endogenous force that keeps valuations stationary when they deviate from steady state—this stability property is key to supporting self-fulfilling fluctuations. For this reason, our Internet Appendix provides three alternatives to the growth-valuation link, each of which also supports self-fulfilling fluctuations.

Redistributive fluctuations. A core message of our analysis is that indeterminacy is most often *redistributive*. That is, the conditions for indeterminacy in the aggregate valuation Q_t are significantly stronger than the conditions for indeterminacy in the cross-section of valuations $(q_{n,t})_{n=1}^N$. For instance, we prove that, if the elasticity of intertemporal substitution (EIS) is less than one, the aggregate valuation is pinned down uniquely in our setup, while the valuation distribution is not. Even if the EIS is above one, but the growth-valuation link is not too powerful, the only possible indeterminacy is redistributive. These results are what justify the paper title referring to "idiosyncratic risk."

A novel prediction is that asset booms are less likely to be synchronized global phenomena and more likely to be found in individual sectors and geographic locations (Brunnermeier and Schnabel, 2015). Instead of being in sync, asset boom-bust cycles should co-move negatively: a crash in one asset market necessarily coincides with a boom in another.

Despite fluctuations being redistributive, the self-fulfilling volatility of our model maintains, under some natural conditions, a systematic factor structure. In particular, we prove that, if sunspot valuation shocks maintain a stable cross-sectional correlation, then there is necessarily a single-factor structure to our idiosyncratic volatility. The existence of this common component to redistributive risk is an intriguing outcome of our model, since it provides a plausible microfoundation to researchers that have modeled exogenously time-varying idiosyncratic volatility and its macroeconomic effects (Di Tella, 2017, 2020; Di Tella and Hall, 2022; Iachan et al., 2022).

Market segmentation. We are particularly interested in the effects of cross-sectional market segmentation. First of all, financial market segmentation is reasonable in many real-world contexts, especially our applications that follow.[1] Second, segmentation enriches our baseline theoretical predictions in several intriguing dimensions.

While the theoretical results above on equilibrium multiplicity and volatility hold even under complete financial markets, layering on some market segmentation introduces *real effects*. Agents in location n must hold a concentrated portfolio of their local equity, and so their consumption responds to self-fulfilling shocks. Again, because our fluctuations are primarily redistributive in nature, the cross-sectional wealth and

[1] For example, there is the well known "home bias" among international asset holdings (French and Poterba, 1991). In the firm context, there is also pervasive evidence that corporate insiders hold concentrated exposures to their own firms, perhaps for incentive reasons (May, 1995; Guay, 1999; Himmelberg et al., 2002; Panousi and Papanikolaou, 2012).

consumption distributions become central objects. For instance, capital flows become intertwined with boom-bust cycles (Caballero et al., 2008): when one location's valuation features a boom that coincides with a fall in another location's valuation, the rising market simultaneously borrows from the falling market.

In the presence of uninsurable consumption fluctuations, agents naturally command risk premia on their local equity, which is exposed to these same shocks. As with the level of self-fulfilling volatility, the associated risk premia also contain a single-factor structure. Thus, our theory sheds light on time-varying compensation for idiosyncratic risk exposure.

Applications. We consider two applications of our framework. First, we interpret our locations as firms, and we interpret the agents in the model as corporate insiders that hold concentrated positions in the firm. With this interpretation, our model produces firm-level idiosyncratic stock returns whose volatility has a factor structure. Because corporate insiders hold undiversified exposures to their own stocks, firm-specific shocks command a risk premium, whose magnitude is a function of the aggregate idiosyncratic volatility factor. These patterns are supported by the empirical finance literature on firm dynamics (Hopenhayn, 1992; Sutton, 1997; Luttmer, 2007; Gabaix, 2009) and firm-specific stock returns (Campbell et al., 2001; Herskovic et al., 2016).

Second, we extend the model to include "non-tradable" consumption goods and interpret our locations as countries. Self-fulfilling volatility in asset prices now spills over into real exchange rates, via capital flows. This volatility is in excess of fundamentals and creates unshared risks, which help resolve various exchange rate puzzles (e.g., the PPP and Backus-Smith puzzles). The paper discusses these patterns in more detail, along with a growing international macro literature that embraces market incompleteness in pursuit of resolutions (Gabaix and Maggiori, 2015; Lustig and Verdelhan, 2019; Itskhoki and Mukhin, 2021).

Contributions to the multiplicity literature. Our construction of self-fulfilling equilibria shares a similar flavor to seminal studies that build sunspot shocks around a stable steady state. We differ from this literature in some of the assumptions we adopt—we require neither overlapping generations (Azariadis, 1981; Cass and Shell, 1983; Farmer and Woodford, 1997) nor aggregate increasing returns (Farmer and Benhabib, 1994) to induce stability. Instead, we provide several new examples of "stabilizing forces." Our equilibrium construction is also more general in permitting an arbitrary number of markets, a broad class of fundamental shocks, and a broad class of self-fulfilling shocks.

A key feature of our analysis is that self-fulfilling fluctuations are less likely to be aggregate phenomena. This result echoes Loewenstein and Willard (2006), who show that noise-trader volatility in De Long et al. (1990) cannot survive the endogeneity of the interest rate in general equilibrium. This result also distinguishes our mechanism from several other studies that build multiplicity through collateral constraints or other financing frictions (Krishnamurthy, 2003; Benhabib and Wang, 2013; Miao and Wang, 2018; Schmitt-Grohé and Uribe, 2021), which continue to operate in single-location, closed-economy settings.

Our results are closer to the OLG model of Gârleanu and Panageas (2020) and the limited enforcement model of Zentefis (2022). Like those models, our multiplicity arises when there are multiple traded assets and a link between valuations and some fundamental. Our contribution is to provide a much more general analysis, explore the consequences of market segmentation, and apply our model to novel applications.

Outline. The remainder of the paper proceeds as follows. Section 1 describes the model. Section 2 analyzes the deterministic equilibria of the model. Section 3 analyzes stochastic complete-markets equilibria. Section 4 layers on some financial market segmentation. Section 5 studies some applications of the model. That section also contains lengthy discussions of the existing literature in the context of each application.

1 Model

An economy is set in continuous time that is indexed by $t \geq 0$.

Endowments. There are N "locations" in the economy. Each location can represent a firm, a sector, an industry, a country, or a distinct financial market. Each location n receives an endowment stream $y_{n,t}$, with the aggregate endowment denoted by $Y_t := \sum_{n=1}^{N} y_{n,t}$. The endowment of location n follows

$$dy_{n,t} = y_{n,t}\Big[g_{n,t}dt + \nu dB_t + \hat{\nu} d\hat{B}_{n,t} - \hat{\nu}\sum_{i=1}^{N}\frac{y_{i,t}}{Y_t}d\hat{B}_{i,t}\Big], \tag{1}$$

where $(B, \hat{B}_1, \ldots, \hat{B}_N)$ is an $(N+1)$-dimensional standard Brownian motion. We think of B as the aggregate fundamental shock and $\hat{B} := (\hat{B}_n)_{n=1}^{N}$ as location-specific fundamental shocks. For simplicity, each location has symmetric shock exposures ν and $\hat{\nu}$. Our results do not rely on the presence of fundamental shocks, and we could very well set $\nu = \hat{\nu} = 0$. In fact, in most of the derivations presented in the body of the paper, we shut down these fundamental shocks for clarity. We leave the local expected growth rate g_n arbitrary for now and discuss this growth rate in more detail below. Summing across n in Eq. (1), the aggregate endowment follows

$$dY_t = Y_t[g_t dt + \nu dB_t]. \tag{2}$$

We have purposefully specified location-specific shock exposures in Eq. (1) in order that the aggregate volatility is the constant ν in Eq. (2).

Financial Markets. Each location offers a single asset in positive net supply that is a claim to its local endowment $y_{n,t}$—we refer to this as the local equity market. The equilibrium equity price in location n is $q_{n,t}y_{n,t}$, where $q_{n,t}$ is the price-dividend ratio. In addition to these N distinct equity markets, there is a risk-free bond in zero net supply that offers equilibrium interest rate r_t. Finally, there is an integrated futures market for trading claims on the fundamental shocks $(B, \hat{B}_1, \dots, \hat{B}_N)$, with each future in zero net supply. Allowing these futures markets is not critical (i.e., our theoretical results would be similar without them), and in fact one may believe it more realistic that risk-sharing of location-specific shocks is imperfect. That said, the inclusion of these futures markets affords theoretical clarity to our results on multiplicity, in the sense that we isolate the minimal needed deviation from perfect markets.

A different representative agent lives in each location. In the first part of the paper (Sections 2-3), we assume markets are *complete*, in the sense that these agents can invest in all local equity markets, the short-term bond market, and the futures markets. This complete-markets case transparently conveys the construction of our equilibrium multiplicity. In the second part of the paper (Sections 4-5), we assume markets are *segmented*, in the sense that representative agent n can only invest in local equity market n, in addition to the bond and futures markets. (Hence, the bond and futures markets are integrated throughout the paper.)

In the complete-markets case, there is a unique stochastic discount factor (state-price density) ξ_t. In the segmented-markets case, each location has a potentially different state-price density $\xi_{n,t}$ (in fact, because this case features incomplete markets, each location could potentially have many state-price densities, but we focus on the one which corresponds to the marginal utility of agent n). To capture both cases at once, we often write $\xi_{n,t}$ for the location-n state-price density.

Budgets and Constraints. Based on the assumptions so far, the financial wealth $w_{n,t}$ of the representative agent in location n evolves as

$$dw_{n,t} = (w_{n,t}r_t - c_{n,t})dt + \vartheta_{n,t}(\eta_t dt + dB_t) + \hat{\vartheta}_{n,t} \cdot (\hat{\eta}_t dt + d\hat{B}_t) \\ + \sum_{i=1}^{N} \theta_{n,i,t}\Big(\frac{1}{q_{i,t}}dt + \frac{d(q_{i,t}y_{i,t})}{q_{i,t}y_{i,t}} - r_t dt\Big), \quad w_{n,0} = q_{n,0}y_{n,0}. \tag{3}$$

The terms $\vartheta_{n,t}$ and $\hat{\vartheta}_{n,t}$ represent positions in the futures markets, which have unit exposure to the shocks $(B, \hat{B})$ and earn those shocks' market prices of risk $(\eta, \hat{\eta})$, to be determined in equilibrium. The term $\theta_{n,i,t}$ is agent n's position in equity market i. In the complete market case, these positions are unrestricted. In the segmented market case, agents face an additional constraint that says $\theta_{n,i,t} = 0$ for all $i \neq n$ (no investment in other equity markets). Note that $w_{n,t} - \sum_{i=1}^{N} \theta_{n,i,t}$ represents the amount of saving (borrowing, if negative) in the bond market. The initial condition $w_{n,0} = q_{n,0}y_{n,0}$ says that the agent's initial endowment is a single share of the local equity, although this does not necessarily pin down their initial wealth, as the price $q_{n,0}$ is endogenous. In addition to Eq. (3), the agent must obey the solvency constraint $w_{n,t} \geq 0$ (this is the natural borrowing limit) and the No-Ponzi condition

$$\lim_{T\to\infty} \xi_{n,T}\Big(w_{n,T} - \sum_{i=1}^{N} \theta_{n,i,T}\Big) = 0. \tag{4}$$

The No-Ponzi condition prohibits asymptotic indebtedness.

Preferences. Agents have infinite lives, CRRA utility with elasticity of intertemporal substitution (EIS) ρ^{-1}, time discount rate $\delta > 0$, and rational expectations. Mathematically, preferences are represented by

$$\mathbb{E}_0\Big[\int_0^\infty e^{-\delta t}\frac{c_{n,t}^{1-\rho} - 1}{1-\rho}dt\Big]. \tag{5}$$

The limiting case $\rho = 1$ corresponds to logarithmic utility, which we use to illustrate many results.

Market Clearing. Clearing of the goods and bond markets is standard: $\sum_{n=1}^{N} c_{n,t} = Y_t$ and $\sum_{n=1}^{N}(w_{n,t} - \sum_{i=1}^{N} \theta_{n,i,t}) = 0$. In addition, all the futures markets need to clear, so $\sum_{n=1}^{N} \vartheta_{n,t} = 0$ and $\sum_{n=1}^{N} \hat{\vartheta}_{n,t} = 0$. Local equity market clearing is $\sum_{i=1}^{N} \theta_{i,n,t} = q_{n,t}y_{n,t}$ for each n. Finally, combining the bond and equity market clearing conditions leads to the convenient aggregate wealth constraint $\sum_{n=1}^{N} w_{n,t} = \sum_{n=1}^{N} q_{n,t}y_{n,t} = Q_tY_t$, where Q_t is the aggregate price-dividend ratio.

Growth Rates. To obtain our interesting multiplicity results, we model a type of endogeneity in fundamental growth rates. We assume local growth rates take the form

$$g_{n,t} = g + \lambda(q_{n,t} - q^*), \quad \lambda \geq 0, \tag{6}$$

for some common parameters g, λ, and q^*. We always take q^* to be the "steady state" valuation ratio, which is a function of the other parameters. The assumption of a linear growth-valuation link is convenient analytically, and in many cases it is without much loss of generality.[2]

Eq. (6) is a reduced-form representation of a microfounded link between dividend growth and asset prices. One microfoundation of this link is that asset prices carry payoff-relevant information. Corporate managers filter this information from stock prices and update their investment decisions accordingly (Chen et al., 2007; Bakke and Whited, 2010; Goldstein and Yang, 2017; Bond et al., 2012). Under this interpretation, $\lambda > 0$ is sensible: when valuations are above their typical level, managers infer positive information and invest more.[3] Internet Appendix C provides three alternatives to the endogenous growth in Eq. (6) that also generate the possibility of non-fundamental volatility—we discuss these alternatives in Section 3.4 in more detail.

Under the linear growth-valuation link (6), the aggregate growth rate is given by

$$g_t := \sum_{n=1}^{N} \frac{y_{n,t}}{Y_t} g_{n,t} = g + \lambda(Q_t - q^*), \tag{7}$$

where recall the aggregate valuation ratio is Q_t. Eq. (7) illustrates the convenience of the linear functional form: aggregate growth only depends on the aggregate valuation, rather than the entire cross-sectional distribution of valuations.

Extrinsic Shocks. To allow the possibility of non-fundamental volatility, conjecture that

[2]Indeed, much of the analysis is confined local to steady state, so any nonlinear growth-valuation link would effectively be linearized anyway. In a previous working paper version, we allowed in many theoretical results an arbitrary nonlinear link $g_{n,t} = \Gamma(q_{n,t})$, for some increasing function $\Gamma(\cdot)$.

[3]While a full microfoundation of this managerial "feedback effects" mechanism is beyond the scope of this paper, a natural question is why the managers in our model would not simply learn their firm's fundamentals from the futures markets. One possibility is simply that the managers do not properly filter out the information in futures markets from their stock prices. Another possibility is that the full set of futures is not traded, particularly with regard to firms' idiosyncratic fundamentals; if so, managers would have to partly learn their idiosyncratic fundamentals from their stock prices, even though these stock prices may also be polluted by non-fundamental dynamics.

the price-dividend ratio of each location's asset follows a stochastic process of the form

$$dq_{n,t} = q_{n,t}\Big[\mu^q_{n,t}dt + \varsigma^q_{n,t}dB_t + \hat{\varsigma}^q_{n,t}\cdot d\hat{B}_t + \sigma^q_{n,t}\cdot dZ_t\Big], \tag{8}$$

where Z is an N-dimensional Brownian motion, independent from B and $\hat{B}$. The shock Z_t is *extrinsic*, and it is the source of self-fulfilling fluctuations, if any exist.

We refer to $\sigma^q_{n,t}$ as the *self-fulfilling volatility* of location n. If $\sigma^q_{n,t} \neq 0$ for some n, we say that the economy exhibits self-fulfilling volatility; if $\sigma^q_{n,t} = 0$ for all n, we say self-fulfilling volatility does not exist.

Economically, the extrinsic Z shocks arise from sources that we do not explicitly model—they are sunspot shocks. In all papers with sunspot shocks, a common question is "what is the sunspot?" We do not take any stand on this, but there are several possibilities explored in the literature. One popular candidate is investor sentiment or signals that coordinate beliefs (Benhabib et al., 2015); other candidates highlighted by the literature are shocks with vanishingly small impacts on fundamentals so that they are effectively extrinsic but still retain a coordination role (Manuelli and Peck, 1992).

No-Bubble Assumption. As a consequence of the No-Ponzi conditions (4) and individual agents' transversality condition $\lim_{T\to\infty}\mathbb{E}_t[\xi_{n,T}w_{n,T}] = 0$, it is possible to show that $\lim_{T\to\infty}\xi_{n,T}q_{n,T}y_{n,T} = 0$ holds in any equilibrium. This is enough for our purposes, but we impose the following slightly stronger "no-bubble" condition for theoretical clarity.

Condition 1. For each n, it holds that $\lim_{T\to\infty}\mathbb{E}_t[\xi_{n,T}q_{n,T}y_{n,T}] = 0$.

Because of Condition 1, equity prices equal present values of future dividends. Self-fulfilling volatility in our model is thus consistent with classical no-bubble theorems (e.g., Santos and Woodford, 1997; Loewenstein and Willard, 2000) that give conditions under which bubbles are not possible.

Equilibrium. This completes the description of the model. An *equilibrium* is a set of adapted processes $(y_{n,t}, c_{n,t}, w_{n,t}, q_{n,t}, \xi_{n,t}, (\theta_{n,i,t})_{i=1}^N, \vartheta_{n,t}, \hat{\vartheta}_{n,t})_{t\geq 0}$ for $1 \leq n \leq N$ and $(r_t, \eta_t, \hat{\eta}_t)_{t\geq 0}$, adapted to the augmented filtration generated by $(B, \hat{B}, Z)$, such that: agents maximize (5) subject to their budget constraint (3), their No-Ponzi condition (4), and their solvency constraint $w_{n,t} \geq 0$; Eqs. (1), (6), and (8) all hold; all markets clear; and Condition 1 holds. In Appendix A, we derive the complete set of conditions characterizing equilibrium that we use going forward. In expositing our results below, we bring forth and explain any critical equations, so it is not necessary for the reader to consult Appendix A unless a detailed derivation is desired.

Endowment and consumption shares. Because of the scalability properties of our model, we repeatedly make use of the endowment and consumption shares to characterize equilibrium:

$$\alpha_{n,t} := \frac{y_{n,t}}{Y_t} \quad \text{and} \quad x_{n,t} := \frac{c_{n,t}}{Y_t}. \tag{9}$$

The dynamics of all stationary variables can be described without reference to Y_t, once we know $(\alpha_{n,t}, x_{n,t})_{n=1}^N$. In Sections 2-3, markets are complete so that $x_{n,t}$ plays no role; but when market segmentation is introduced in Sections 4-5, the consumption distribution becomes important.

2 Multiplicity of deterministic equilibria

To get to the core forces quickly, we start with the deterministic equilibria of our model. Let us shut down all fundamental shocks ($\nu = \hat{\nu} = 0$), and let us examine equilibria with $\varsigma^q_{n,t} = 0$, $\hat{\varsigma}^q_{n,t} = 0$, and $\sigma^q_{n,t} = 0$. These equilibria highlight most of the intuition that is also present in the more complex stochastic equilibria to come.

2.1 Derivation of equilibrium

In a deterministic equilibrium, each location's equity is priced according to the following Euler equation:

$$\frac{\dot{q}_{n,t}}{q_{n,t}} + g_{n,t} + \frac{1}{q_{n,t}} = r_t. \tag{10}$$

At the same time, aggregating optimal consumption dynamics $\dot{c}_{n,t}/c_{n,t} = \rho^{-1}(r_t - \delta)$ across locations, we obtain the equilibrium interest rate

$$r_t = \delta + \rho g_t. \tag{11}$$

Substituting (11) into (10) and using the expressions for the growth rates $g_{n,t}$ and g_t, we obtain

$$\frac{\dot{q}_{n,t}}{q_{n,t}} + \frac{1}{q_{n,t}} = \delta + (\rho - 1)\Big(g - \lambda q^* + \lambda Q_t\Big) - \lambda\Big(q_{n,t} - Q_t\Big). \tag{12}$$

From Eq. (12), we see that the steady state of this economy features $q^* = \frac{1}{\delta + (\rho-1)g}$ and hence $g_{n,t} = g$ for all n. (Consequently, we always implicitly assume $\delta + (\rho - 1)g > 0$ so that a steady-state equilibrium exists.)

Eq. (12) suggests both the mathematics and the intuition for how the growth-valuation link matters for determinacy. Consider the log case ($\rho = 1$), and imagine Q_t is fixed at steady state $q^* = \delta^{-1}$. Then, the dynamical system for $q_{n,t}$ becomes

$$\dot{q}_{n,t} = -1 + q_{n,t}(\delta + \lambda q^*) - \lambda q_{n,t}^2.$$

This dynamical system has two steady states, but only the one with $q_n = q^*$ is relevant (because that one coincides with the aggregate valuation). As long as $\lambda > \delta/q^* = \delta^2$, this larger steady state is locally stable, in the sense that $\frac{\partial \dot{q}_n}{\partial q_n}\big|_{q_n = q^*} = \delta - \lambda q^* < 0$. The left panel of Figure 1 plots the dynamical system for various values of λ. When the economy has this stability property, equilibrium is indeterminate: one may start with any $q_{n,0}$ close enough to q^*, and the valuation drifts towards q^*.

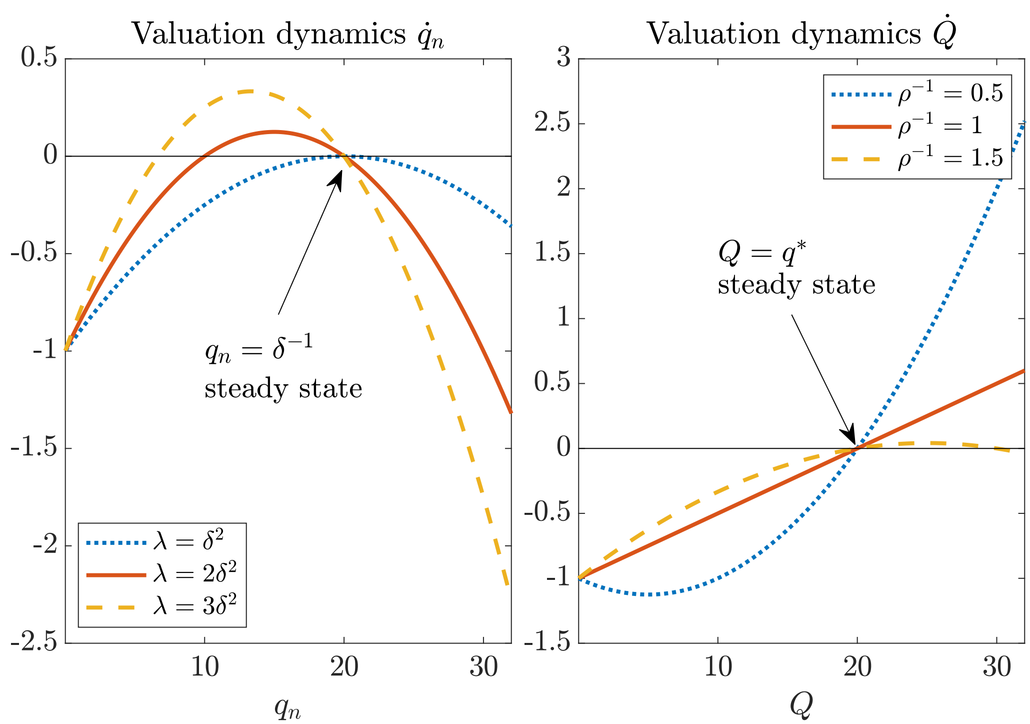

Figure 1: Valuation dynamics.

Notes. The left panel plots the dynamics of a single location's valuation with unitary EIS ($\rho^{-1} = 1$) and various levels of λ. The right panel plots the aggregate valuation dynamics with $\lambda = 2\delta^2$ and various levels of EIS ρ^{-1}. Both panels assume $\delta = 0.05$ and $g = 0$.

Why is the arbitrary initial valuation $q_{n,0}$ self-fulfilled? Intuitively, if the asset valuation is low, then the growth-valuation link induces growth to be low as well. Low growth is disappointing for investors, whose required return of r_t must instead be met by capital gains. In other words, $\dot{q}_{n,t}/q_{n,t}$ must rise to satisfy investors—this force brings valuations back up towards steady state. An analogous argument holds if $q_{n,0}$ takes any value slightly above steady state.

The point of the ensuing analysis in this section is to generalize these arguments. We would like to consider how the EIS ρ^{-1} matters and to understand the consequences of local indeterminacy on the aggregate valuation ratio Q_t. We provide a complete answer to these questions.

Let us now compute the dynamics of the aggregate valuation ratio Q_t. From its definition, we have

$$\dot{Q}_t = \sum_{n=1}^{N} (\dot{\alpha}_{n,t} q_{n,t} + \alpha_{n,t} \dot{q}_{n,t})$$

The dynamics of endowment shares are given by

$$\dot{\alpha}_{n,t} = \alpha_{n,t}(g_{n,t} - g_t). \tag{13}$$

Then, using Eqs. (10), (11), and (13), we obtain

$$\dot{Q}_t = -1 + \Big[\delta + (\rho - 1)(g - \lambda q^*)\Big] Q_t + \lambda(\rho - 1) Q_t^2. \tag{14}$$

Similar to the location-specific valuations, we may compute the local stability of the aggregate valuation. Notice that $\frac{\partial \dot{Q}}{\partial Q}\big|_{Q=q^*} = \delta + (\rho - 1)g + \lambda(\rho - 1)q^*$. If $\rho \geq 1$, then the steady state is unstable, in the sense that $\frac{\partial \dot{Q}}{\partial Q}\big|_{Q=q^*} > 0$. This instability suggests there can be no indeterminacy in Q_t, which must always equal its steady-state value q^*. Conversely, if $\rho < 1$, then the dynamics of Q_t are stable if λ is large enough. The right panel of Figure 1 plots the aggregate valuation dynamics for various levels of ρ^{-1}. Thus, it seems that the EIS is critical for whether there can be *aggregate indeterminacy*.

The intuition for why the EIS affects the nature of indeterminacy—i.e., whether valuations in aggregate can be indeterminate or not—is as follows. A higher conjectured aggregate valuation Q_t leads to a higher aggregate growth rate g_t, through Eq. (7). Higher aggregate growth increases the demand for borrowing and consumption today, which raises the interest rate r_t because current aggregate output is pre-determined at Y_t. Whereas higher aggregate growth g_t works to raise Q_t through a "cash flow effect", the rise in r_t works to offset this and lower Q_t through a "discount rate effect"; the balance of these effects controls whether or not aggregate indeterminacy can arise. If the EIS is high, a small rise in r_t suffices to induce savings and equilibrate markets, and so the discount rate effect is small, and Q_t can rise in a self-fulfilled way. If the EIS is low, however, the rise in r_t must be more significant, and so the conjectured boom in Q_t cannot be self-justified. This feedback through the interest rate is what starkly distinguishes the questions of aggregate indeterminacy from that in the local economies.

2.2 General classification of equilibria

Let us now provide a general result. Staring at Eqs. (12) and (14), we see that this constitutes an autonomous dynamical system for $(q_{n,t})_{n=1}^N$ and Q_t. This dynamical system is nonlinear, but we may linearize it near steady state to evaluate its stability properties, which is the key criterion for whether or not equilibrium indeterminacy exists.

The equilibrium vector is $\mathbf{q}_t := (q_{1,t}, \ldots, q_{N,t}, Q_t)'$, and so local stability properties are determined via the eigenvalues of the $(N+1) \times (N+1)$ Jacobian matrix

$$J := \left[\frac{\partial \dot{\mathbf{q}}_t}{\partial \mathbf{q}_t'} \Big|_{ss} \right].$$

In the appendix, we solve $Jv = \eta v$ to compute the eigenvalues and eigenvectors of J. It turns out that J has two eigenvalues,

$$\begin{aligned} \eta_- &= \delta + (\rho - 1)g - \lambda q^* \\ \eta_+ &= \delta + (\rho - 1)g + (\rho - 1)\lambda q^*, \end{aligned}$$

with the corresponding eigenvectors

$$\begin{aligned} v(\eta_+) &= \mathbf{1}_{N+1} \\ v(\eta_-) &\in \{e_1, \ldots, e_N\}, \end{aligned}$$

where e_n is the nth elementary vector. In other words, the eigenvalue η_- has multiplicity N. Using this result, the appendix shows that the various asset prices can be written, close to steady state, as

$$\begin{aligned} q_{n,t} &\approx q^* + (q_{n,0} - Q_0)e^{\eta_- t} + (Q_0 - q^*)e^{\eta_+ t}, \quad n = 1, \ldots, N; \\ Q_t &\approx q^* + (Q_0 - q^*)e^{\eta_+ t} \end{aligned}$$

From these standard results, we simply examine how the various parameters influence the signs of η_- and η_+ to prove the following theorem. (All proofs are in Appendix B.)

Theorem 1. *Consider deterministic equilibria. Then,*

(i) *Suppose* $\lambda > [\delta + (\rho - 1)g]^2 > (1 - \rho)\lambda$, *so that* $\eta_- < 0 < \eta_+$. *Then, any equilibrium local to steady state must have* $Q_0 = q^*$, *but* $(q_{n,0})_{n=1}^N$ *can deviate locally from steady state in arbitrary directions that satisfy* $q^* = \sum_{n=1}^N \alpha_{n,0} q_{n,0}$.

(ii) *Suppose* $(1 - \rho)\lambda > [\delta + (\rho - 1)g]^2$, *so that* $\eta_- < \eta_+ < 0$. *Then,* Q_0 *and* $(q_{n,0})_{n=1}^N$ *can all deviate locally from steady state in arbitrary directions that satisfy* $Q_0 = \sum_{n=1}^N \alpha_{n,0} q_{n,0}$.

Theorem 1 allows us to make three central points that we have already hinted at. First, if the growth-valuation link is strong enough, $\lambda > [\delta + (\rho - 1)g]^2$, then the steady state is locally stable, which permits some amount of indeterminacy. Quantitatively, the required connection between growth rates and valuations is not too extreme: if the steady-state valuation ratio is $q^* = 25$, then growth rates must be at least 0.4% above average when valuations are 10% above average.[4] Second, if the EIS is smaller than or equal to one, $\rho^{-1} \leq 1$, then any indeterminacy is purely *redistributive indeterminacy* in the sense that the aggregate valuation ratio cannot deviate from steady state. Redistributive indeterminacy means that if some locations' valuations are high, then other locations' valuations must be low. Third, if the EIS is larger than one, $\rho^{-1} > 1$, and the strength of growth-valuation link is sufficiently high, $\lambda > \frac{[\delta + (\rho - 1)g]^2}{1 - \rho}$, then even the aggregate valuation ratio can be indeterminate.

Figure 2 displays the indeterminacy regions implied by Theorem 1. In making the plot, λ and ρ are allowed to take various values, but q^* is held fixed. (Note that, unless $g = 0$, q^* changes with ρ. So implicitly we are varying δ along with ρ in order to keep q^* fixed. One can think of this as "recalibrating" the primitive model parameters to match a given observed valuation ratio.)

In this paper, we are particularly interested in the redistributive indeterminacy, for a few reasons. First of all, only redistributive indeterminacy can exist if the EIS is below one. While there is significant debate on the magnitude of the EIS, we view EIS below one as plausible.[5] Second, even if the EIS is above one, aggregate indeterminacy requires a significantly stronger growth-valuation link than is required to produce redistributive indeterminacy (e.g., with $\rho^{-1} = 2$, the growth-valuation link must be twice as strong to induce aggregate indeterminacy). While no evidence exists directly measuring the

[4] For a $(100 \times p)\%$ higher valuation, the growth rate is higher by $\lambda((1 + p)q^*) - \lambda(q^*) = p\lambda q^* > p/q^*$, where the last inequality uses the requirement $\lambda > (q^*)^{-2}$. For a 10% higher valuation ($p = 0.1$) with $q^* = 25$, we have $p/q^* = 0.004 = 0.4\%$. More generally, the semi-elasticity $\frac{dg_{n,t}}{d \log q_{n,t}}$ must at least be $1/q^*$.

[5] For instance, micro evidence such as Campbell and Mankiw (1989) suggests an EIS significantly below one, whereas some macro-finance evidence stemming from the literature on "long-run risks" beginning with Bansal and Yaron (2004) point to an EIS above one. Still other studies that consider heterogeneity, such as Guvenen (2009) and Gârleanu and Panageas (2015), suggest significant heterogeneity in EIS but do not require calibrations of the EIS above one to match aggregate asset-price data.

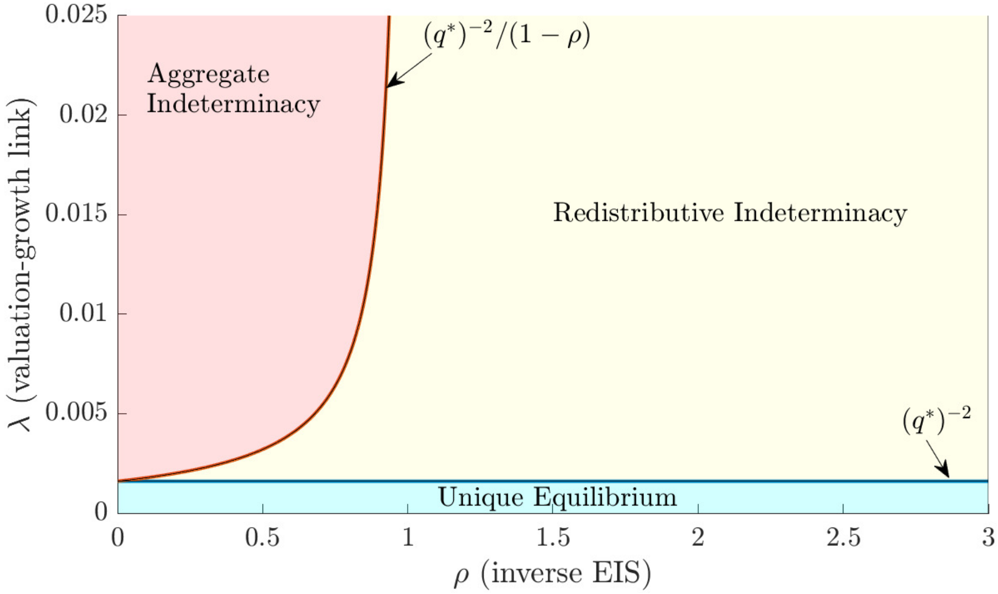

Figure 2: Indeterminacy Regions.

Notes. When $(1-\rho)\lambda > (q^*)^{-2}$, "Aggregate Indeterminacy" is possible, in the sense that Q is not pinned down. When $(1-\rho)\lambda < (q^*)^{-2} < \lambda$, only "Redistributive Indeterminacy" is possible, in the sense that Q is pinned down, but $(q_n)_{n=1}^N$ are not. When $\lambda < (q^*)^{-2}$, a "Unique Equilibrium" results. The plot uses $q^* = 25$.

magnitude of the growth-valuation link, we think too large of a link is less plausible. In sum, redistributive indeterminacy exists under much broader conditions than aggregate indeterminacy, and so we view it as more likely (see also Figure 2).

Finally, it is very clear from this deterministic environment (which necessarily has complete financial markets) that market segmentation is not critical to asset-price indeterminacy. Here, all agents consume multiples of each other and yet asset prices are indeterminate. The same is true in the next section, where we introduce volatility but maintain complete markets.[6]

3 Stochastic equilibria under complete markets

In this section, we want to generalize the indeterminacy results of Section 2 by allowing for self-fulfilling stochastic fluctuations. First, we generalize the claim that redistributive multiplicity is, in many cases, the only type of multiplicity (i.e., when the EIS is below

[6]The reader may expect the First Welfare Theorem to hold with complete markets, and if so, solving the model via a "planner's problem" would generically result in a unique outcome. So how could indeterminacy emerge? Intuitively, one can understand our growth-valuation link as a pecuniary externality. Such externalities cause violations of the First Welfare Theorem and allow equilibrium non-uniqueness.

one or when the EIS is above one but the growth-valuation link is insufficiently strong). Second, we provide a general construction of redistributive stochastic fluctuations to highlight the factor structure in volatility. And finally, we provide conditions under which such a construction constitutes an equilibrium.

3.1 Prevalence of redistribution

Let us first generalize the claim that redistributive indeterminacy is the "more common" type of indeterminacy in this model. In particular, any indeterminacy is necessarily redistributive when $\rho \geq 1$, and a local version of this result also holds when $\rho < 1$ if additionally the growth-valuation link is insufficiently strong. To provide a transparent derivation, assume the absence of fundamental shocks ($\nu = \hat{\nu} = 0$)—this is generalized in the formal results below.

With complete markets, there is perfect consumption risk-sharing, so no agent retains exposure to extrinsic shocks. In particular, the Euler equation $\dot{c}_{n,t}/c_{n,t} = \rho^{-1}(r_t - \delta)$ still holds here. A first implication of this perfect risk sharing is the absence of non-fundamental risk premia: the pricing equation for local equity is now the analogous expression (recall $\mu^q_{n,t}$ is the geometric drift of $q_{n,t}$)

$$\mu^q_{n,t} + g_{n,t} + \frac{1}{q_{n,t}} = r_t. \tag{15}$$

A second implication of perfect risk sharing is the lack of any precautionary savings due to extrinsic shocks, and so the equilibrium interest rate is

$$r_t = \delta + \rho g_t, \tag{16}$$

exactly as in a deterministic equilibrium. For these reasons, much of the analysis of Section 2 carries over to this section.

In particular, the *valuation drifts* remain identical to the deterministic case. Substituting the expressions for r_t in (16) and growth rates $g_{n,t}$ and g_t in (6) and (7) into the pricing equation (15), we have

$$\mu^q_{n,t} + \frac{1}{q_{n,t}} = \delta + (\rho - 1)\Big(g - \lambda q^* + \lambda Q_t\Big) - \lambda\Big(q_{n,t} - Q_t\Big). \tag{17}$$

Using (13), (17), and the definition of Q_t, the aggregate valuation ratio satisfies

$$dQ_t = \Big[-1 + \Big(\delta + (\rho - 1)(g - \lambda q^*) \Big) Q_t + \lambda(\rho - 1) Q_t^2 \Big] dt + \sigma_t^Q \cdot dZ_t, \tag{18}$$

where $\sigma_t^Q := \sum_{n=1}^N \alpha_{n,t} q_{n,t} \sigma_{n,t}^q$. As before, Q_t still has unstable dynamics when $\rho \geq 1$, and so all indeterminacy is redistributive. The following lemma provides a general result that also allows for fundamental shocks.[7]

Lemma 1. *Suppose financial markets are complete. Suppose $\rho \geq 1$. Then, the only bounded solution for the aggregate valuation is $Q_t = q^*$ forever.*

Using a very similar methodology, but also restricting attention to all possible equilibria where Q_t does not exceed steady state by too much, we may also provide a counterpart to Lemma 1 for $\rho < 1$ and for a sufficiently tame growth-valuation link.

Lemma 2. *Suppose financial markets are complete. Suppose $\rho < 1$, and $\lambda < (\frac{1}{1+\varepsilon})^2 (\frac{1}{1-\rho}) (\frac{1}{q^*})^2$, for some $\varepsilon > 0$. Then, among equilibria where $Q_t \leq q^*(1+\varepsilon)$ forever, the only solution for the aggregate valuation is $Q_t = q^*$ forever.*

3.2 General construction of redistributive fluctuations

If Q_t is constant, any indeterminacy is purely redistributive. Here, we flesh out the implications of redistributive fluctuations—the analysis of this section applies even beyond complete financial markets, and is also used later when markets are segmented.

In particular, constant aggregate valuations require, from Eq. (18),

$$\sum_{n=1}^N \alpha_{n,t} q_{n,t} \sigma_{n,t}^q = 0. \tag{19}$$

In other words, extrinsic shocks must offset across local prices. There are an infinite number of choices for $(\sigma_{n,t}^q)_{n=1}^N$ that satisfy (19). The general solution is as follows. Let

[7] Eq. (18) is a one-dimensional backward stochastic differential equation (BSDE). One solution is $Q_t = (\delta + (\rho - 1)g)^{-1}$. Lemma 1 uses standard mathematical results on uniqueness of solutions to infinite-horizon BSDEs to prove that this is the only solution. Essentially, these BSDE tools generalize to stochastic environments the idea that unstable dynamics induce unique solutions. Note that Lemma 1 relies on a linear growth-valuation link, because otherwise the dynamics of Q_t would depend on the entire distribution of valuations $(q_{n,t})_{n=1}^N$. Although we see no clear reason why this would modify the result that redistribution is necessary, the analysis of a multi-dimensional BSDE system is substantially more complex than the one-dimensional case (especially when, as we expect to be the case for our model, the distribution of valuations is indeterminate even when the aggregate valuation is pinned down uniquely).

M_t be any $N \times N$ matrix-valued process with unit length columns and $\text{rank}(M_t) < N$. Then, for an arbitrary vector v_t in the null-space of M_t, set

$$\begin{bmatrix} \alpha_{1,t} q_{1,t} \sigma^q_{1,t} & \alpha_{2,t} q_{2,t} \sigma^q_{2,t} & \cdots & \alpha_{N,t} q_{N,t} \sigma^q_{N,t} \end{bmatrix} = M_t \text{diag}(v_t). \tag{20}$$

Every collection of diffusions $(\sigma^q_{n,t})_{n=1}^N$ that solve Eq. (19) must take the form (20) for some M_t and v_t. However, this solution is a bit too general to be useful. By appealing to a few sensible properties, that in particular restrict M_t, we aim to characterize a "broad class" of redistributive equilibrium volatilities.

Assumption 1. *In equilibria satisfying Eq. (19), equivalently Eq. (20), we assume that $M_t \equiv M$ is constant over time and satisfies $\text{rank}(M) = N - 1$.*

Assumption 1 restricts M_t in two ways. First, setting $M_t \equiv M$ to be a constant matrix equivalently restricts the cross-sectional price correlations to be constant—it will be clear soon that $\text{corr}_t[d \log q_{i,t}, d \log q_{j,t}] = (M_t e_i)' M_t e_j$. The idea here is that coordination determines these cross-sectional correlations, and it seems sensible and more sustainable for such coordination to be relatively anchored over time. As the simplest way to capture such anchored correlation, we restrict $M_t \equiv M$ to be a constant matrix.

Second, we only consider matrices with one degree of degeneracy. This restriction is justified by the following mathematical property: among all possible choices of M, those with $\text{rank}(M) = N - 1$ are of "full measure" in the sense that a random singular matrix would have a rank of $N - 1$ with probability 1. Intuitively, one can imagine agents trying to coordinate on a volatile equilibrium; almost-surely they will coordinate on one where $\text{rank}(M) = N - 1$. For this reason, assuming $\text{rank}(M) = N - 1$ is really a generic property of our volatile equilibria.

Let us now explain how to construct all possible volatile and redistributive equilibria. More specifically, the following procedure can be used to construct every possible solution to Eqs. (19)-(20) that satisfies Assumption 1.

Lemma 3. *Consider the following procedure:*

1. *Pick any non-negative, non-zero $N \times 1$ vector v^* with unit length. Set the matrix M to any $N \times N$ matrix with $null(M) = span(v^*)$.*

2. *Let $(\psi_t)_{t \geq 0}$ be any non-negative scalar process.*

3. *Given $(\alpha_{n,t}, q_{n,t})_{n=1}^N$ at time t, set*

$$\sigma^q_{n,t} = \psi_t \frac{v^*_n}{\alpha_{n,t} q_{n,t}} M e_n, \tag{21}$$

where v_n^* is the nth element of v^*, and e_n is the nth elementary vector.

Then, $(\sigma_{n,t}^q)_{n=1}^N$ solves Eq. (19) for each t. Furthermore, every solution to Eq. (19) that also satisfies Assumption 1 can be constructed by the above procedure.

Remark 1 (Redistributive SDE system)**.** Lemma 3 defines volatilities in terms of other endogenous objects, which may confuse some readers as to how exactly this is a "construction." The way to think about Lemma 3 is as a construction of an SDE system for $(\alpha_{i,t}, q_{i,t})_{i=1}^N$. To see this, note that Eq. (17) already defines the drift $\mu_{n,t}^q$ given the time-t state $(\alpha_{i,t}, q_{i,t})_{i=1}^N$. Then, Eq. (21) defines the diffusion $\sigma_{n,t}^q$ given the time-t state $(\alpha_{i,t}, q_{i,t})_{i=1}^N$ and the auxiliary state ψ_t. Altogether, we have a *forward SDE system* which can thus be simulated. The only free inputs into this SDE system are the vector v^* and process ψ_t, which are not yet restricted. The requirements of equilibrium place some restrictions on ψ_t, which are discussed in Section 3.3.

The key implication of Lemma 3 is a *single-factor structure* in volatilities. Indeed, note that the level of volatility in each location is given by

$$\|\sigma_{n,t}^q\| = \psi_t \frac{v_n^*}{\alpha_{n,t} q_{n,t}}. \tag{22}$$

These volatilities feature a scalar process ψ_t that moves all volatilities up and down together. As explained by the lemma, this is a necessary outcome: every set of redistributive volatilities satisfying Assumption 1 has such a structure. Whereas it is often difficult to pinpoint generic predictions in models of multiple equilibria, this single-factor volatility structure prevails in almost all (in the precise mathematical sense) redistributive sunspot equilibria with constant cross-location correlations. (So as to avoid overreaching with this implication, we note that relaxing the constant correlation structure would introduce additional volatility factors. In principle, our model could feature nearly arbitrarily time-varying correlations, and so it is certainly true that our common-factor implication is sensitive to disciplining these correlations somehow.)

3.3 Verifying that redistributive fluctuations is an equilibrium

Lemma 3 essentially constructs a forward SDE system for $(\alpha_{n,t}, q_{n,t})_{n=1}^N$ by setting the diffusion coefficients explicitly in terms of an auxiliary process ψ_t and vector v^*. As discussed already, this SDE system necessarily features redistributive fluctuations. The remaining question is which choices of ψ_t and v^* constitute an equilibrium? Can we have $\psi_t > 0$ at some times, so that there exists self-fulfilling volatility?

To answer this question, we appeal to stability considerations: as long as we construct volatilities in a way that keeps valuations non-explosive, we have an equilibrium. How can one ensure non-explosion? Start from Eq. (17), and substitute $Q_t = q^*$ and the diffusion (21), to get the following dynamic equation for local valuations:

$$dq_{n,t} = \Big[-1 + q_{n,t}\Big(\delta + (\rho - 1)g - \lambda(q_{n,t} - q^*)\Big)\Big]dt + \psi_t \frac{v_n^*}{\alpha_{n,t}}(Me_n) \cdot dZ_t, \tag{23}$$

The key issue for equilibrium is whether the dynamics in (23) keep $q_{n,t}$ from hitting zero or from exploding towards infinity (and thereby violating some Ponzi condition). Luckily, the drift of $q_{n,t}$ in (23) is identical to the deterministic equilibrium case, so we expect the stability properties to carry over here. The following proposition settles how ψ_t can be chosen to ensure equilibrium (generalized to allow fundamental shocks).

Proposition 1. *Suppose financial markets are complete. Suppose $\lambda > (q^*)^{-2}$. Assume either $N \geq 3$ or $\hat{v} = 0$. Then, an equilibrium exists with redistributive self-fulfilling volatility, which can be constructed as follows. Set all valuation diffusions $(\sigma^q_{n,t})_{n=1}^N$ according to Lemma 3, but where $(\psi_t)_{t\geq 0}$ is a non-negative process that additionally satisfies the following two properties:*

(P1) $\psi_t / \min_n \alpha_{n,t}$ is bounded;

(P2) ψ_t vanishes as $\min_n q_{n,t}$ approaches $\frac{1}{q^*}(\epsilon + \lambda^{-1})$ from above, for $0 < \epsilon < (q^*)^2 - \lambda^{-1}$, or as $\max_n q_{n,t}$ approaches Kq^* from below, for some $K > 1$.

Such a process $(\psi_t)_{t\geq 0}$ exists, and in this construction, we have $\frac{1}{\lambda q^*} < q_{n,t} \leq Kq^*$ for all t, almost-surely.

Proposition 1 proves the existence of a large class of equilibria with self-fulfilling volatility, indexed by the scalar process ψ_t. The amount of volatility is only restricted by the requirements (P1) and (P2), which say that volatility vanishes "far from steady state" (P2) and that all volatilities stay bounded (P1). If so, then volatility never pushes valuations outside of their "stable region" which ensures that no explosion or free disposal condition is violated.[8]

[8]Technically, the proof of existence relies on so-called "stochastic stability" tools (e.g., Khasminskii, 2011), which require a different analysis than the deterministic stability tools used to prove Theorem 1. In particular, as we have already mentioned, Lemma 3 constructs an SDE system for $\{(\alpha_{n,t}, q_{n,t})_{n=1}^N : t \geq 0\}$. We want to prove a solution exists to this SDE system such that all the valuations remain in some domain U; in particular, valuations must remain positive (to satisfy free disposal) and bounded above (which will be needed to prove the No-Ponzi and No-Bubble conditions hold). Under very general conditions (e.g., boundedness of diffusion coefficients) a solution to the SDE system exists up to an exit time from U—this is the easy step. The critical step is that prescriptions (P1)-(P2) ensure that ψ_t vanishes appropriately at the domain's boundary ∂U, which allows us to prove, via stochastic stability theorems, the exit time from U is almost-surely unbounded. Khorrami and Mendo (2024) also makes use of similar tools to prove the existence of sentiment-driven equilibria.

While conditions (P1)-(P2) involve endogenous objects, it is straightforward to pick an ψ_t that satisfies these conditions, precisely because the SDE for $(\alpha_{i,t}, q_{i,t})_{n=1}^N$ is a *forward SDE*. For instance, consider the following explicit construction of ψ_t satisfying (P1)-(P2). Let $\bar{\psi}_t$ be *any* scalar process. Pick any number $L > 0$. Define the endogenous variable $\underline{\psi}_t := \min_n \alpha_{n,t} \wedge (Kq^* - \max_n q_{n,t}) \wedge (\min_n q_{n,t} - \frac{\epsilon+\lambda^{-1}}{q^*})$. Then, $\psi_t = \bar{\psi}_t \wedge L\underline{\psi}_t$ satisfies (P1)-(P2), and it is clear that it can be constructed at time t given knowledge of the system $(\alpha_{i,t}, q_{i,t})_{n=1}^N$. All we are doing in this example is forcing ψ_t to vanish if the system ever deviates too far from steady state or if volatilities ever grow too large.

Remark 2 (Aggregate sunspots)**.** The remainder of the paper is primarily focused on redistributive volatility, where Q_t is not subject to sunspot shocks. That said, it is possible to construct an example where Q_t also has sunspot volatility, as long as the EIS is larger than one ($\rho^{-1} > 1$) and the growth-valuation link is sufficiently strong ($\lambda > \frac{1}{(1-\rho)(q^*)^2}$). Internet Appendix D contains a formal result and example construction.

Remark 3 (Other indeterminacies)**.** This section and main results have been presented focusing on indeterminacy of the sunspot volatilities $(\sigma_{n,t}^q)_{n=1}^N$. However, by following the logic closely, the reader can rightly guess that there is also an indeterminacy on how the local valuations $q_{n,t}$ load on the fundamental shocks; that is, $(\varsigma_{n,t}^q)_{n=1}^N$ and $(\hat{\varsigma}_{n,t}^q)_{n=1}^N$ are also indeterminate. For example, in a redistributive equilibrium, fundamental exposures would be subject to conditions analogous to Eq. (19), e.g., $\sum_{n=1}^N \alpha_{n,t} q_{n,t} \varsigma_{n,t}^q = 0$ for the exposures to the aggregate shock dB_t. Besides satisfying this redistribution condition, the exposures could take nearly arbitrary values cross-sectionally. If we further impose a constant-correlation assumption (as in Assumption 1), redistributive aggregate exposures would necessarily take the form $\varsigma_{n,t}^q = \psi_t \frac{v_n^*}{\alpha_{n,t} q_{n,t}}$ for some scalar process ψ_t and some vector v^*. This paper does not explore these possibilities in more detail for two reasons: (i) we view it as simpler and theoretically clearer to study indeterminacies via an extrinsic shock; and (ii) redistribution of fundamental exposures is required under identical conditions as redistribution of sunspot exposures.

3.4 Alternative sources of endogeneity and stability

By now, it should be clear that endogenous growth rates are essential. Having understood that the role of endogenous growth is to induce stable dynamical systems, a natural question is whether alternative sources of endogeneity might work similarly. Internet Appendix C provides three additional examples of endogeneity that also work as "stabilizing forces."

In Internet Appendix C.1, we show that valuation-dependent beliefs can create a stable dynamical system and hence support self-fulfilling volatility. In particular, we suppose that, while true growth rates remain constant, investors become more optimistic about growth when valuations rise. Perhaps agents use asset prices to construct their beliefs about growth to simplify a complex underlying filtering problem, or perhaps rising asset prices just create euphoria amongst investors. Either way, such optimism about growth encourages asset demand which fulfils the initial conjecture of a higher valuation. This specification mirrors our baseline model's growth-valuation link, but only in investors' heads. An interesting outcome is that beliefs are endogenously extrapolative (Barberis et al., 2015).

In Internet Appendix C.2, we show that under-investment, of the type induced by "debt overhang" (e.g., Hennessy, 2004; DeMarzo et al., 2012), creates the needed stability. The main idea is that potential gains from investment are high relative to actual investment, which leaves some surplus on the table. As prices rise and boost investment, debt overhang problems shrink, and some of this surplus is captured by local investors. The extra returns gained this way compensate investors for lower dividend yields and ensure stable price-dividend ratios. An intriguing implication is that under-investment can be a self-fulfilling phenomenon for reasons other than those previously identified (e.g., non-convex technologies or borrowing constraints).

In Internet Appendix C.3, we show that an overlapping generations economy with "creative destruction" (e.g., Aghion and Howitt, 1992; Gârleanu and Panageas, 2020) also produces the required stability. Creative destruction here is represented as new firms entering and displacing incumbents. If the amount of creative destruction is itself a function of asset prices, high asset prices can be self-fulfilled by a reduction in new firm entry, and vice versa. High valuations reduce dividend yields to investors, but living cohorts are compensated with the preservation of their firms, which removes the need for valuations to continue growing and thus creates stability.

Economically, Eq. (6) and the examples in Internet Appendix C share a common property: when valuations rise so that dividend yields fall, investors are compensated somehow. This compensation can take the form of higher dividend growth rates, higher perceived growth rates, a drop in under-investment wedges, or less creative destruction. It is likely that many other examples of stabilizing forces also exist. By identifying several, we stress that a wide range of plausible environments all generate a similar type of stability that can support self-fulfilling volatility.

4 Segmented markets

While the previous sections with complete markets demonstrated transparently how to detect indeterminacies and construct self-fulfilling volatility, we are particularly interested in a situation where markets are segmented. We begin with an example construction of an equilibrium with self-fulfilling volatility. Then, we explore some key properties, including the effects of price volatility on consumption, risk premia, and the bond market.

4.1 Construction: log utility and self-fulfilling volatility

Let us first generalize the construction of self-fulfilling fluctuations to an environment with market segmentation. Because the analysis becomes substantially more complex with segmentation, we assume from here on that $\rho = 1$ (i.e., log utility).

The first property that carries over to this environment is *redistribution*. Investors with log utility consume a fraction δ of their wealth, so the aggregate wealth-consumption (price-dividend) ratio is $Q_t = \delta^{-1}$. Therefore, any self-fulfilling volatility is necessarily redistributive across locations, and the volatility construction of Lemma 3 continues to hold. Due to analytical complexity, we do not prove the necessity of redistribution for the non-log case, but we would expect this to be true (i.e., we expect versions of Lemmas 1-2 to carry over to the segmented-markets setting).[9]

The next proposition provides a segmented-markets counterpart to the complete-markets existence and characterization result from Proposition 1. The upshot is that, as before, a large class of equilibria exists with self-fulfilling volatility, indexed by the single volatility factor ψ_t.

Proposition 2. *Suppose $\rho = 1$ and $\lambda > \delta^2$. Assume either $N \geq 3$ or $\hat{v} = 0$. Then, an equilibrium exists with redistributive self-fulfilling volatility, which can be constructed as follows. Set all valuation diffusions $(\sigma^q_{n,t})_{n=1}^N$ according to Lemma 3, but where $(\psi_t)_{t\geq 0}$ is a non-negative process that additionally satisfies the following two properties:*

(P1) $\psi_t / \min_n \alpha_{n,t}$ *is bounded and* $\psi_t^2 < \delta^{-1} \min_n x_{n,t}$;

[9] Intuitively, we expect such results to hold, because the complete- and segmented-markets model dynamics coincide when endogenous volatilities "vanish far from steady state." Ultimately, the stability or instability properties of Q_t dynamics when such volatilities vanish are what determines whether or not there can be aggregate indeterminacy or not. However, proving this formally is substantially more difficult with segmentation because the BSDE for the aggregate valuation now depends on the full cross-section of endowment shares, consumption shares, wealth-consumption ratios, and asset valuations.

(P2) ψ_t vanishes as $\min_n q_{n,t}$ approaches $\delta(\epsilon + \lambda^{-1})$ from above, for $0 < \epsilon < \delta^{-2} - \lambda^{-1}$, or as $\max_n q_{n,t}$ approaches $K\delta^{-1}$ from below, for some $K > 1$.

Such a process $(\psi_t)_{t\geq 0}$ exists, and in this construction, we have $\frac{\delta}{\lambda} < q_{n,t} \leq K\delta^{-1}$ for all t, almost-surely.

Remark 4 (Bond market)**.** While we assume local equity markets are segmented, we do require some amount of integration. In particular, the *bond market must remain integrated* for our multiplicity results. One obvious way to see this is to imagine a single location living in autarky (i.e., both equity and bond markets are segmented from all other locations). That would correspond to a single-location model ($N = 1$), and we already know such an economy cannot exhibit indeterminacy when $\rho = 1$.

Another way to understand the importance of the bond market is to think through the mechanics of equilibrium. If the valuation $q_{1,t}$ increases to an extrinsic shock, agent 1 has higher future endowments via the growth-valuation link. Knowing her future endowments will be higher, it is optimal to consume now. But her local endowment $y_{1,t}$ has not changed in the short run; to consume in excess of her endowment—i.e., to consume $c_{1,t} > y_{1,t}$—she must borrow from other locations. The reverse holds for agent 2 who supplies funds to the bond market, due to a reduction in his local valuation ratio: his future endowments are lower, which incentivizes savings to smooth consumption. Without the bond market, no valuation changes could be justified.

Remark 5 (Partial equity-issuance)**.** Our equity markets are completely segmented, but this is not essential. Indeed, imagine agent n could issue equity to outsiders, up to maximum of $\phi q_{n,t} y_{n,t}$. Then, local investors still must retain a fraction $1 - \phi$ of their local equity shocks, which is enough to create the phenomena we discuss below—self-fulfilling consumption fluctuations, risk premia, and precautionary savings demand.

4.2 Real effects and risk premia

So far, the equilibria with segmented markets are similar to those with complete markets: self-fulfilling fluctuations exist, are characterized by a single factor, and are redistributive across markets.

The novelty under segmented markets is that each agent n is exposed to non-tradable shocks, through the extrinsic shocks hitting their local asset price. Two consequences arise: (i) self-fulfilling asset-price volatility has *real effects* by creating fluctuations in the cross-sectional consumption distribution; (ii) agents now command *risk premia* as compensation for self-fulfilling fluctuations.

The argument is as follows. In segmented markets, agent n must hold the entirety of asset n, so price shocks hit her wealth. With log utility, these wealth shocks transmit one-for-one to consumption. Therefore, redistribution in asset valuations causes consumption redistribution. Furthermore, because marginal utility fluctuates with consumption, agents necessarily demand risk compensation for their sunspot exposures.

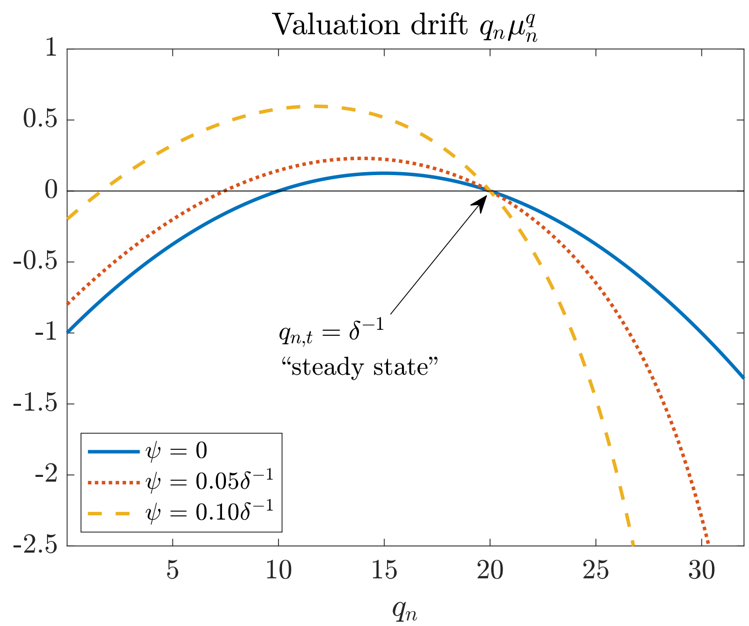

Figure 3: Valuation dynamics with segmented equity markets.

Notes. Parameters are $\rho = 1$, $\delta = 0.05$, $\lambda = 2\delta^2$, and $\nu = \hat{\nu} = 0$. The economy has $N = 2$ locations, and features $v^* = (1,1)'/\sqrt{2}$. Finally, for plotting purposes, the right panel assumes $x_{n,t} = \alpha_{n,t} = 0.5$ for each n (consumption and endowment shares coincide).

Figure 3 plots the expected capital gains $q_{n,t}\mu^q_{n,t}$ in a "symmetric" stochastic equilibrium example with $N = 2$ locations. (In particular, we assume M and v^* are such that a single extrinsic shock redistributes across the two locations.) The different values of ψ correspond to different levels of volatility, since recall $\alpha_{n,t} q_{n,t} \|\sigma^q_{n,t}\| = \psi_t$. For $\psi = 0$ (solid line), dynamics are identical to the deterministic equilibrium. For $\psi > 0$, the presence of volatility steepens the drift, because low-valuation locations have higher return volatility and thus higher risk premia. Risk premia must be met by higher expected capital gains, so this force increases $\mu^q_{n,t}$ when $q_{n,t}$ is low, and vice versa when $q_{n,t}$ is high.

Indeed, the formula for the valuation drift without fundamental shocks and with log utility ($\rho = 1$) is

$$q_{n,t}\mu^q_{n,t} = \underbrace{-1 + \delta(1+\lambda/\delta^2)q_{n,t} - \lambda q^2_{n,t}}_{\text{deterministic component}} + \underbrace{\delta\frac{(v^*_n\psi_t)^2}{\alpha_{n,t}x_{n,t}}}_{\text{risk premium}} - \underbrace{q_{n,t}\delta^2\sum_{i=1}^{N}\frac{(v^*_i\psi_t)^2}{x_{i,t}}}_{\text{precautionary savings}} \tag{24}$$

where recall $x_{n,t}$ is the location-n consumption share. (The general formula for $\mu^q_{n,t}$ is in Eq. (A.2) of Appendix A.) The term labeled "deterministic component" is the entire drift when $\psi_t = 0$ and is identical to $\dot{q}_{n,t}$ in Eq. (10). The term labeled "risk premium" arises because investor n demands compensation for the self-fulfilling volatility in his local equity, a risk premium which must be delivered via future capital gains. We elaborate in detail on term labeled "precautionary savings," which arises from the equilibrium interest rate, in Section 4.3 below. To understand the steepening effect that $\psi > 0$ has in Figure 3, simply observe that $q_{n,t}$ scales the precautionary savings term, so that tends to dominate the risk premium term when $q_{n,t}$ is high, and vice versa.

4.3 Precautionary savings and the bond market

How does self-fulfilling volatility feed back into the bond market? The equilibrium interest rate of our model is given by

$$r_t = \underbrace{\delta + \rho g_t - \frac{1}{2}\rho(\rho+1)\nu^2}_{\substack{\text{representative-agent}\\\text{interest rate}}} - \underbrace{\frac{1}{2}\rho(\rho+1)\sum_{n=1}^{N}x_{n,t}\|\sigma^c_{n,t}\|^2}_{\substack{\text{idiosyncratic}\\\text{precautionary savings}}} \tag{25}$$

If all locations were perfectly integrated, a representative agent would exist and the equilibrium interest rate would be $\delta + \rho g_t - \frac{1}{2}\rho(\rho+1)\nu^2$, which reflects discounting plus growth minus the precautionary savings motive due to aggregate volatility.

If locations are segmented, and self-fulfilling volatility takes hold, then an additional precautionary savings term arises. In particular, $\|\sigma^c_{n,t}\|$ is agent n's consumption growth exposure to extrinsic shocks. Consumption growth is exposed to extrinsic shocks because local equity is exposed and agents cannot share this risk with other locations. Such risk is *idiosyncratic*, because it necessarily aggregates to zero across locations (i.e., $\sum_{n=1}^{N} x_{n,t}\sigma^c_{n,t} = 0$, because aggregate consumption Y_t is not exposed to extrinsic shocks). As in classical models of exogenous idiosyncratic risks, all agents want to save to self-insure against this idiosyncratic risk, which has the effect of reducing r_t (Bewley, 1986; Huggett, 1993; Aiyagari, 1994).

In our log utility ($\rho = 1$) example construction from Proposition 2, the idiosyncratic precautionary savings term becomes

$$\sum_{n=1}^{N} x_{n,t} \|\sigma_{n,t}^{c}\|^2 = \sum_{n=1}^{N} \frac{1}{x_{n,t}} (\delta \alpha_{n,t} q_{n,t})^2 \|\sigma_{n,t}^{q}\|^2 = (\delta \psi_t)^2 \sum_{n=1}^{N} \frac{(v_n^*)^2}{x_{n,t}}$$

A rise in the volatility factor ψ_t increases all agents' idiosyncratic risks, which increases the precautionary savings motive.

The poorest agents (i.e., locations with low $x_{n,t}$) have the highest marginal utility and are thus most sensitive to a rise in volatility. In equilibrium, these poor agents decrease their consumption to pay down existing debt balances as ψ_t rises, while richer agents consume more today by reducing their savings. To see this dynamic, examine the expected consumption growth rate of each location in equilibrium:

$$\mu_{n,t}^{c} = r_t - \delta + \nu^2 + (\frac{\delta \psi_t v_n^*}{x_{n,t}})^2$$

(This is simply agent n's Euler equation, with extrinsic consumption volatility $\sigma_{n,t}^{c}$ substituted in.) If ψ_t rises, consumption growth rises in poor locations (those with small $x_{n,t}$) and falls in rich locations (high $x_{n,t}$). As suggested earlier, this happens because poor locations strongly increase their precautionary savings when idiosyncratic risk rises.

5 Applications

In this section, we discuss two applications of self-fulfilling volatility in segmented markets. The first considers "locations" to be firms and explores the growth and risk premium consequences of excess idiosyncratic volatility. The second application interprets "locations" as countries in an international macroeconomy, which features excess volatility of exchange rates and can speak to some international finance puzzles. For all results of this section, we assume consumers have log utility ($\rho = 1$).

5.1 Firm-specific risks and undiversified insiders

In this section, we interpret each "location" n as a firm, and "representative investor" n as the corporate insiders of that firm (e.g., CEOs). With the model applied to firms, many microfoundations of a growth-valuation link seem plausible. Endogenous cash flow growth rates can be thought of here as "feedback effects" between stock prices and investment (Bond et al., 2012). Alternatively, as discussed in Internet Appendix C.2, one could consider firms with debt outstanding, in which case debt-overhang problems lead to a connection between valuations and investment efficiency. Either interpretation seems appropriate for firms, and both foster self-fulfilling volatility.

Our segmentation assumptions also seem plausible in this application. In fact, firm insiders are often not fully diversified (May, 1995; Guay, 1999; Himmelberg et al., 2002; Panousi and Papanikolaou, 2012) and their individual preferences and other characteristics seem to matter in firms' decision processes (Bertrand and Schoar, 2003; Graham et al., 2013). Such concentrated risk exposure can arise as an optimal pay-for-performance compensation contract in the presence of moral hazard or signalling/selection issues (Holmström, 1979; Leland and Pyle, 1977; Rock, 1986). Our model partly captures this phenomenon. We say "partly" because our investors have access to a futures market that allows them to share risks from the location-specific fundamental shocks $d\hat{B}_t$. If we wanted to better capture a setting of corporate insiders, we could also eliminate this particular futures market, in which case the insiders would effectively be holding a portfolio of their firm's equity along with outside borrowing/lending (position in riskless bonds) and trading in the aggregate stock market index (futures on dB_t).[10]

Self-fulfilling volatility is in many cases redistributive, in that it aggregates to zero. Yet this idiosyncratic volatility features a *common component*: firm-n self-fulfilling return volatility is $\|\sigma^q_{n,t}\| = \psi_t v^*_n / \alpha_{n,t} q_{n,t}$, which scales with the common factor ψ_t. (Recall: the single-factor structure comes from assuming a stable cross-correlation structure, and then examining the "full measure" of remaining volatile equilibria.) In the data, Campbell et al. (2001) and Herskovic et al. (2016) document a significant and highly time-varying common component in idiosyncratic return volatility.

Not only should idiosyncratic stock volatilities contain a common factor, fundamentals should too. Indeed, firms that are doing particularly well in the stock market should also have particularly high investment and growth rates. Firms doing poorly should be "underinvesting." This spread between firm-level growth rates is also magnified by the

[10] In such an extension, the key modification would be that fundamental idiosyncratic risk demands a risk premium from undiversified insiders. Mathematically, the absence of a futures market for $d\hat{B}_t$ implies $\hat{\vartheta}_{n,t} = 0$ for each n, which implies insider consumption growth has the following exposure to $d\hat{B}_t$:

$$\frac{1}{dt}\mathrm{Cov}_t[\frac{dc_{n,t}}{c_{n,t}}, d\hat{B}_t] = \frac{\delta \alpha_{n,t} q_{n,t}}{x_{n,t}}(\hat{v}_{n,t} + \hat{\varsigma}_{n,t}).$$

(See Eq. (A.12) in Appendix A with $\hat{\vartheta}_{n,t} = 0$ substituted.) Consequently, the expected capital gain in Eq. (A.2) must be augmented by the risk premium from this exposure, namely $\frac{\delta \alpha_{n,t} q_{n,t}}{x_{n,t}} \|\hat{v}_{n,t} + \hat{\varsigma}_{n,t}\|^2$. This modification would substantially complicate the type of equilibrium construction done in Proposition 2, because now the dynamics of $q_{n,t}$ become coupled with those of $\alpha_{n,t}$ and $x_{n,t}$, even when valuation volatility vanishes. However, the spirit of the analysis does not change.

common volatility factor ψ_t.

The firm dynamics literature (Hopenhayn, 1992; Sutton, 1997; Luttmer, 2007; Gabaix, 2009) has emphasized random log-normal growth (plus a "friction") as a possible reason for the fat-tailed firm size distribution. One quantitative difficulty has been explaining the thickness of the tail with realistic levels of firm-specific volatility. Our framework can alleviate this issue, since larger firms tend to grow faster. A positive correlation between size and growth rates magnifies the size dispersion in any real variable such as sales.

Although it is redistributive, self-fulfilling volatility commands a risk premium, because insiders hold concentrated, undiversified exposures to their own stocks. While the other implications above would hold even without this concentrated exposure, segmentation is required for this risk premium implication. From Eq. (24), the idiosyncratic risk premium for firm-n equity is given by

$$\text{risk premium} = \frac{\delta(\psi_t v_n^*)^2}{x_{n,t}\alpha_{n,t}q_{n,t}}.$$

When self-fulfilling volatility spikes (ψ_t rises), measured risk premia also rise. In the data, Herskovic et al. (2016) find that the common component in idiosyncratic volatility is priced, consistent with this implication. (In our model, shocks to ψ_t are not necessarily priced without further restrictions, which is why we only say "consistent" with our risk premium implication. In particular, for ψ-shocks to be priced, we would need firms with higher self-fulfilling risk premia to also have larger exposures to $d\psi_t$.)

The expression for the idiosyncratic risk premium also hints at how certain stock market anomalies may be related to our mechanisms. The risk premium is higher for firms with low valuations (so-called value firms with low $q_{n,t}$) and low market cap (so-called small firms with low $\alpha_{n,t}q_{n,t}$)—see Fama and French (1992). Given the vast amount of research on these issues, we do not overemphasize this connection, but it is interesting that it emerges naturally from our framework.

To get a rough sense of the dynamics, Figure 4 presents results from a simulation exercise with $N = 500$ firms. Our example contains a spatial component to sunspot correlations, embedded in M, whereby "nearby firms" are more highly correlated, whereas firms further apart can even be negatively correlated. "Nearby firms" can be interpreted as firms in similar geographic locations, similar industries, or connected via a production network. Despite the correlation structure, the sunspot shocks are idiosyncratic or redistributive in the sense that they aggregate to zero. In particular, we set M by a "circulant

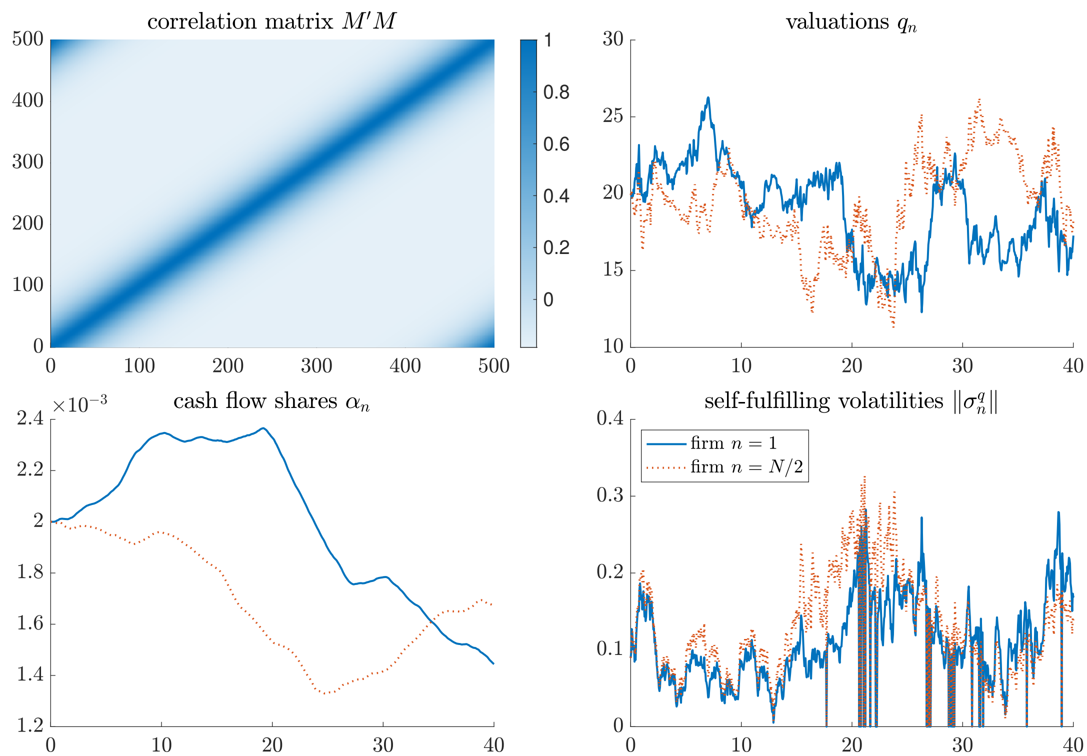

Figure 4: Dynamics from a simulation of $N = 500$ firms.

Notes. The simulation begins in the symmetric steady state: $\alpha_n = x_n = 1/N$ and $q_n = \delta^{-1}$ for all n. The volatility process ψ_t is then set as follows. Let Ψ_t be a Feller square-root process driven by the aggregate fundamental shock:

$$d\Psi_t = -\rho_\Psi(\bar{\Psi} - \Psi_t)dt + \sigma_\Psi\sqrt{\Psi_t}dB_t.$$

Along the simulation, define the object

$$\iota_t := \mathbf{1}\Big\{\max_n q_{n,t} > 2\delta^{-1}\Big\} + \mathbf{1}\Big\{\min_n q_{n,t} < \frac{\delta}{\lambda} + 2\Big\} + \mathbf{1}\Big\{\min_n \alpha_{n,t} < \frac{1}{10N}\Big\} + \mathbf{1}\Big\{\min_n x_{n,t} < \frac{1}{100N}\Big\}$$

Put

$$\psi_t = \begin{cases} \sqrt{\Psi_t}, & \text{if } \iota_t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Parameters: $\delta = 0.05$, $g = 0.02$, $\lambda = 0.0075$, $\nu = 0.02$, $\hat{\nu} = 0$, $\bar{\Psi} = 0.0064$, $\sigma_\Psi = -0.08$, $\rho_\Psi = 0.9$.

matrix" with the following column n

$$Me_n \propto \begin{bmatrix} m_{N-n+2} & m_{N-n+3} & \cdots & m_N & m_1 & m_2 & \cdots & m_{N-n+1} \end{bmatrix}' - \bar{m},$$

where $m_n = \zeta^{n-1}$ and $\bar{m} = \frac{1}{N}\sum_{i=1}^N m_i$. The subtraction of $\bar{m}$ from each column ensures that M has zero row-sums. This matrix has rank $N-1$ and a null-space spanned by $v^* = \mathbf{1}'_N/\sqrt{N}$. Figure 4 uses $\zeta = 0.95$. The top left panel graphically illustrates the correlation structure induced by this particular choice of M.

The remaining three panels of Figure 4 display the paths for two firms whose sunspot correlations are negative. Notice how, despite the inverse correlation observed between these locations' valuations, their sunspot volatilities $\|\sigma_n^q\|$ co-move strongly. This property emerges because the scalar factor ψ_t affects all firms' volatilities.

To solidify the quantitative implications, we simulate this economy 1000 times and compute several statistics. Results are displayed in Table 1. Consistent with the magnitudes displayed in Figure 4, a significant amount of volatility is possible—the overall average volatility is 9.3% per annum. This leads to a significant idiosyncratic portion of valuation dynamics. Indeed, doing a principal components analysis (PCA) of valuation changes $(dq_{n,t})_{n=1}^N$, we find that the first PC explains 23.1%. Consequently, even valuation levels $(q_{n,t})_{n=1}^N$ have a weak factor structure, though slightly stronger than changes, with the first PC explaining 44.6%. By contrast, idiosyncratic volatility $(\|\sigma_{n,t}^q\|)_{n=1}^N$ exhibits a strong factor structure—the first PC of the volatility panel explains 68.7% of overall variation.

	Average volatility $\mathbb{E}[\frac{1}{N}\sum_{n=1}^N \|\sigma_{n,t}^q\|]$	1st PC (dq) $\frac{\lambda_1^{dq}}{\sum_{i=1}^N \lambda_i^{dq}}$	1st PC (q) $\frac{\lambda_1^{q}}{\sum_{i=1}^N \lambda_i^{q}}$	1st PC (σ^q) $\frac{\lambda_1^{\sigma}}{\sum_{i=1}^N \lambda_i^{\sigma}}$
mean	0.093	0.231	0.446	0.687
(st.dev.)	(0.012)	(0.033)	(0.083)	(0.070)

Table 1: Statistics from 1000 simulations of the $N = 500$ firm economy. Each simulation has $T = 40$ years and is designed as detailed in the caption of Figure 4. In the final three columns, λ_i^{dq}, λ_i^q, and λ_i^σ denote the ith largest eigenvalues of the variance-covariance matrix of $(dq_{n,t})_{n=1}^N$, $(q_{n,t})_{n=1}^N$, and $(\|\sigma_{n,t}^q\|)_{n=1}^N$, respectively.

5.2 International macro and exchange rates

Our next application interprets "locations" as countries. In this context, there are several plausible justifications for our growth-valuation link, or the related endogeneity mechanisms discussed in Internet Appendix C. Many mechanisms that work at the more micro level also aggregate to the country level. For instance, an entire country can have extrapolative beliefs about their growth rate (Internet Appendix C.1) or macro-level debt overhang problems (Internet Appendix C.2). Second, to engender self-fulfilling volatility, the creative destruction version of the model (Internet Appendix C.3) only requires displacement risk *within a country* and as a function of the local economy valuation—this is a plausible description of how entrepreneurship works, given that the outside option is another activity within the same country.

Partial equity market segmentation is also a reasonable assumption in international finance, and several studies have argued it can potentially speak to some puzzling observations (Gabaix and Maggiori, 2015; Lustig and Verdelhan, 2019; Itskhoki and Mukhin, 2021). We discuss how our model, simply through non-fundamental fluctuations in asset prices, can help address excess exchange rate volatility (e.g., the PPP puzzle), international risk-sharing puzzles (e.g., Backus-Smith puzzle), and co-movements with capital flows and growth.

To tailor our model to the international setting, we introduce a non-tradable endowment $\hat{y}_{n,t}$. For simplicity and theoretical clarity on what is new with our framework, we assume $\hat{y}_{n,t}$ follows the same time-series growth process as the tradable $y_{n,t}$ in Eq. (1); in particular, let $\hat{y}_{n,t} = \kappa y_{n,t}$ for all n. The representative agent of country n consumes $\hat{c}_{n,t}$ of the non-tradable, which in equilibrium is $\hat{c}_{n,t} = \hat{y}_{n,t}$. The tradable market still clears globally via $\sum_{n=1}^{N} c_{n,t} = \sum_{n=1}^{N} y_{n,t} = Y_t$. Agents have preferences over a Cobb-Douglas aggregate of tradables and their local non-tradable, i.e.,

$$\mathbb{E}_0\Big[\int_0^\infty e^{-\delta t}\Big(\phi \log(c_{n,t}) + (1-\phi)\log(\hat{c}_{n,t})\Big)dt\Big]. \tag{26}$$

We set the tradable good as the numéraire, so let $p_{n,t}$ be the relative price of the country n non-tradable. We let $q_{n,t}$ still be the local valuation ratio, so that the total value of the local endowment is $q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})$. Finally, we continue to assume a growth-valuation link according to the linear functional form (6), so that the country n output growth rate is $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$.

This non-tradables setting is identical to Backus and Smith (1993) and many other studies. The solution is as follows. In this model, the consumption basket and price index of country n are given by

$$C_{n,t} := c_{n,t}^{\phi}\hat{c}_{n,t}^{1-\phi}$$
$$P_{n,t} := \frac{c_{n,t} + p_{n,t}\hat{c}_{n,t}}{C_{n,t}}.$$

The total expenditure of country n is thus $P_{n,t}C_{n,t}$. Because of log utility, agents optimally spend δ fraction of their wealth, so

$$P_{n,t}C_{n,t} = \delta w_{n,t}. \tag{27}$$

Cobb-Douglas period utility implies the optimal expenditure shares of tradables and non-tradables are ϕ and $1-\phi$, respectively:

$$c_{n,t} = \phi P_{n,t} C_{n,t} \quad \text{and} \quad p_{n,t}\hat{c}_{n,t} = (1-\phi) P_{n,t} C_{n,t}. \tag{28}$$

Using Eqs. (27)-(28) and non-tradable market clearing $\hat{c}_{n,t} = \hat{y}_{n,t}$, the price index can be written

$$P_{n,t} = \phi^{-1}\Big(\frac{c_{n,t}}{\hat{y}_{n,t}}\Big)^{1-\phi} = \phi^{-\phi}\Big(\frac{\delta w_{n,t}}{\hat{y}_{n,t}}\Big)^{1-\phi}.$$

The *real exchange rate* $\mathcal{E}_t^{i,j}$ between countries i and j, defined as the ratio of their price indexes, is

$$\mathcal{E}_t^{i,j} := \frac{P_{j,t}}{P_{i,t}} = \Big(\frac{x_{j,t}/x_{i,t}}{\hat{y}_{j,t}/\hat{y}_{i,t}}\Big)^{1-\phi}, \tag{29}$$

where $x_{i,t} := c_{i,t}/Y_t$ is the tradable consumption share of country i (because of log utility, $x_{i,t}$ is equivalently the wealth share of country i). When $\mathcal{E}_t^{i,j}$ rises, we say that the country-j exchange rate appreciates.

The remainder of equilibrium, derived in detail in Internet Appendix E, is similar to the baseline model without non-tradables. In particular, there exist non-fundamental equilibria in which the valuation ratios $(q_{n,t})_{n=1}^N$ are hit by sunspot fluctuations that necessarily redistribute across countries (although note that existence of a sunspot equilibrium with non-tradables requires absence of idiosyncratic fundamental shocks $\hat{v} = 0$). We continue to refer to ψ_t as the corresponding volatility factor. The key complication with non-tradables is that local equity is a claim to both tradable and non-tradable output, and the latter's relative price fluctuates endogenously with the other equilibrium objects. This endogeneity couples the dynamics of $(\hat{y}_{n,t}, q_{n,t})_{n=1}^N$, which must now be analyzed as a $2N$-dimensional system to establish stability properties.

The sunspot equilibria of this model, together with the market segmentation assumptions, are helpful in resolving some exchange rate puzzles. The critical element is the dynamics of wealth shares $x_{n,t}$:

$$\frac{dx_{n,t}}{x_{n,t}} = (\delta\psi_t)^2\Big[\Big(\frac{v_n^*}{x_{n,t}}\Big)^2 - \sum_{i=1}^N x_{i,t}\Big(\frac{v_i^*}{x_{i,t}}\Big)^2\Big]dt + \delta\psi_t\Big(\frac{v_n^*}{x_{n,t}}\Big)(Me_n)\cdot dZ_t. \tag{30}$$

In a segmented-markets sunspot equilibrium, $(x_{n,t})_{n=1}^N$ has additional volatility: extrinsic shocks *redistribute wealth across countries*. If markets were integrated, then $(x_{n,t})_{n=1}^N$ would be constant over time. (In any complete-markets, symmetric preference model, the wealth distribution is constant.) We first describe the puzzles and qualitatively how a stochastic wealth distribution provides partial resolution. Then, we present a numerical illustration.

Volatility puzzles. If markets were integrated, wealth share volatility would be absent, and exchange rates would be driven exclusively by endowment shocks. The volatility of exchange rates would be equal to the volatility of output, reflecting the classic volatility puzzles of Meese and Rogoff (1983) and Mussa (1986).[11] Under segmentation, sunspot shocks can inject additional volatility into exchange rates, via the wealth shares $(x_{n,t})_{n=1}^{N}$:

$$\mathrm{Var}[\log \mathcal{E}_t^{i,j}] = \underbrace{(1-\phi)\mathrm{Cov}\Big[\log \mathcal{E}_t^{i,j}, \log \frac{\hat{y}_{j,t}}{\hat{y}_{i,t}}\Big]}_{= \mathrm{Var}[\log \mathcal{E}_t^{i,j}] \text{ with complete markets}} + \underbrace{(1-\phi)\mathrm{Cov}\Big[\log \mathcal{E}_t^{i,j}, \log \frac{x_{j,t}}{x_{i,t}}\Big]}_{\text{excess volatility in segmented markets}}. \tag{31}$$

Our model can sustain equilibria where the majority of exchange rate volatility stems from the wealth distribution. More specifically, the success of this model is its ability to generate significant tradable consumption-share volatility, even when endowments are relatively smooth, and this consumption-share volatility spills into exchange rates.

Risk-sharing puzzles. Our equilibria also help break a tight positive link between exchange rates and relative aggregate consumptions across countries, providing some resolution to the Backus and Smith (1993) puzzle (see also Kollmann, 1995 and Corsetti et al., 2008). To develop the puzzle and its resolution, notice that the relative aggregate consumptions can be written

$$\frac{C_{i,t}}{C_{j,t}} = \left(\frac{c_{i,t}}{c_{j,t}}\right)^{\phi}\left(\frac{\hat{c}_{i,t}}{\hat{c}_{j,t}}\right)^{1-\phi} = \left(\frac{x_{i,t}}{x_{j,t}}\right)^{\phi}\left(\frac{\hat{y}_{i,t}}{\hat{y}_{j,t}}\right)^{1-\phi}. \tag{32}$$

In response to endowment shocks, $C_{i,t}/C_{j,t}$ and $\mathcal{E}_t^{i,j}$ move in the same direction. In response to wealth share shocks, $C_{i,t}/C_{j,t}$ and $\mathcal{E}_t^{i,j}$ move in opposite directions. Under complete markets, only the endowment shocks matter and so $C_{i,t}/C_{j,t}$ and $\mathcal{E}_t^{i,j}$ become perfectly positively correlated. (In this complete-market case, our model has $\mathcal{E}_t^{i,j} = C_{i,t}/C_{j,t}$, a particularly stark representation of Backus and Smith, 1993.) Under segmented markets, wealth share shocks necessarily reduce this correlation; if extrinsic volatility ψ_t is sufficiently high, $C_{i,t}/C_{j,t}$ and $\mathcal{E}_t^{i,j}$ can even become negatively correlated.

[11] Meese and Rogoff (1983) show that the nominal exchange rate is significantly more volatile than macroeconomic aggregates like consumption and output, while Mussa (1986) shows that the real and nominal exchange rate behaviors are tightly linked. See also the survey in Rogoff (1996) on the Purchasing Power Parity (PPP) puzzle.

Capital flows. Financial market integration implies capital flows are irrelevant. And indeed, the complete-markets version of our model has exchange rates pinned down by endowments, independent of any capital flows. But empirically, capital inflows appear to co-move with, and possibly cause, exchange rate appreciations (Froot et al., 2001; Evans and Lyons, 2002; Froot and Ramadorai, 2005; Hau and Rey, 2006; Combes et al., 2012; Camanho et al., 2022; Lilley et al., 2022).

Our segmented-markets equilibria predict a positive link between capital flows and exchange rates, as in Gabaix and Maggiori (2015). An extrinsic shock that raises $x_{n,t}$ is necessarily accommodated by a capital flow into country n from the rest of the world, in order that tradable consumption $c_{n,t}$ can rise above the local tradable endowment $y_{n,t}$. In other words, capital inflows and tradable consumption increases are the same phenomenon in our model. Most importantly, the link to exchange rates is in line with data: a capital inflow causes an appreciation of country n's exchange rate (i.e., $\mathcal{E}_t^{i,n}$ is increasing $x_{n,t}$). One can think of capital inflows as a "demand shock" where domestic agents desire and achieve more tradable consumption, which pushes up prices of non-tradable goods, and hence pushes up the country's real exchange rate.

Growth rates. So far, our discussion has been based on non-fundamental volatility and segmented markets. But the non-fundamental volatility is only possible due to endogenous growth and the growth-valuation link. Suppose $\mathcal{E}_t^{i,n}$ rises due to an extrinsic shock that raises country-n consumption through capital inflows. Recall that this also raises the country-n stock market valuation $q_{n,t}$, which then feeds back into a higher growth rate $g_{n,t}$. Thus, our model predicts that inflows and exchange-rate appreciations positively forecast future growth.

Some attention has been given to the possibility that longer-term growth prospects may drive exchange-rate variation (Colacito and Croce, 2011, 2013; Colacito et al., 2018). This existing "long-run risks" literature has mostly considered a global growth factor, potentially with international heterogeneity in exposures. A key difference in our framework is the emphasis on *relative growth rates* between countries, rather than a common global growth factor. An additional theoretical difference is that the long-run risks literature requires recursive preferences with particular parameter constellations (i.e., EIS greater than one and risk aversion above the inverse EIS), whereas our mechanisms hold for a larger class of preferences. (That said, these types of preferences help deliver non-trivial carry trade expected returns and uncovered interest parity deviations. In unreported calculations, we have computed carry trade profits in our model and found them to be tiny, presumably as a result of log preferences.)

Numerical example. We present a two-country example to highlight some dynamics and magnitudes. With $N = 2$, it is without loss of generality to consider a single sunspot shock (i.e., Z_1). Figure 5 shows the results from one simulation of this economy.

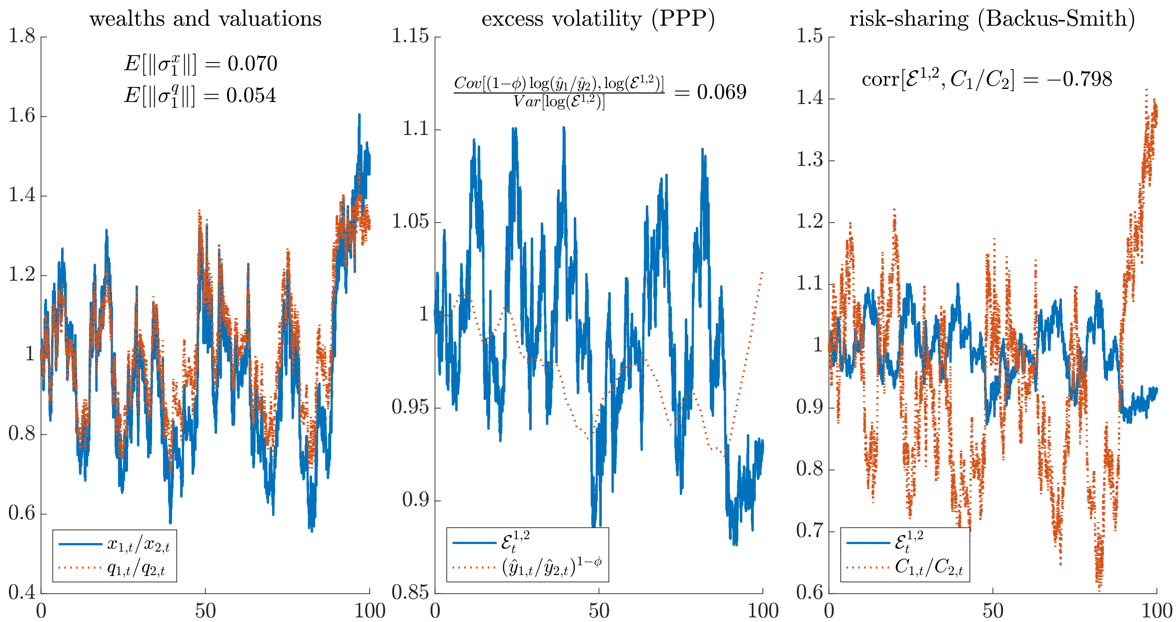

Figure 5: Dynamics from a simulation of $N = 2$ countries.

Notes. The sunspot shock exposure matrix is set so that there is one relevant sunspot shock (i.e., $Z_{1,t}$). In particular, we set $M = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right]$, which has null-space $v^* = (1,1)'/\sqrt{2}$. The volatility process ψ_t is then set as follows. Let Ψ_t be a Feller square-root process driven by the country-1 shock:

$$d\Psi_t = -\rho_\Psi(\bar{\Psi} - \Psi_t)dt + \sigma_\Psi\sqrt{\Psi_t}dZ_{1,t}.$$

Along the simulation, define the object

$$\iota_t := \mathbf{1}\Big\{\max_n(q_{n,t}/q^* - 1) > 0.2\Big\} + \mathbf{1}\Big\{\min_n(1 - q_{n,t}/q^*) > 0.2\Big\} + \mathbf{1}\Big\{\delta \geq -0.001 + \lambda\delta^{-1}\sum_{n=1}^{2}\Phi_{n,t}(1 - \alpha_{n,t})\Big\}$$

Put

$$\psi_t = \begin{cases} \sqrt{\Psi_t}, & \text{if } \iota_t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Parameters: $\delta = 0.05$, $g = 0.02$, $\lambda = 0.0067$, $\phi = 0.75$, $\nu = 0$, $\hat{\nu} = 0$, $\bar{\Psi} = 1$, $\sigma_\Psi = -1$, $\rho_\Psi = 1.5$.

The left panel of Figure 5 illustrates sunspot volatility: both the wealth ratio x_1/x_2 and the valuation ratio q_1/q_2 display significant time-variation. That said, the amount of volatility is not extreme: $\log x_1$ and $\log q_1$ have average annualized volatilities of 0.07 and 0.054, respectively. The middle panel reveals the consequences for the excess volatility of the exchange rate, which varies substantially more than the relatively smooth endowments. In fact, in a standard variance decomposition, endowments only explain about 6.9% of the variance of the exchange rate, with the rest coming from the wealth distribution. Recall that the explanatory power of endowments would be 100% in the complete-markets version of the economy. Finally, the right panel displays the strong negative co-movement between the exchange rate and the consumption ratio, in stark contrast to the Backus-Smith benchmark of perfect positive correlation. Relative to the data, this negative co-movement is actually too strong; this particular simulated path is a bit extreme relative to the set of possible simulations.

To solidify some quantitative magnitudes, we perform 1000 simulations of this economy. The key statistics are displayed in Table 2. Across many simulations, the amount of wealth share volatility is quite modest on average (2.8% p.a.). Still, in line with the PPP puzzle, endowments explain a small fraction of exchange rate dynamics (14.9%). Finally, consistent with Backus-Smith, exchange rates and consumption ratios are only moderately correlated (44.5%). The main takeaway is that adding a small amount of extrinsic wealth distribution volatility can create significant excess exchange rate fluctuations and substantially reduce markers of international risk-sharing.

	Volatility $\mathbb{E}[\|\sigma^x_{1,t}\|]$	PPP covariance $\frac{\mathrm{Cov}[(1-\phi)\log(\hat{y}_1/\hat{y}_2),\log(\mathcal{E}^{1,2})]}{\mathrm{Var}[\log(\mathcal{E}^{1,2})]}$	Backus-Smith correlation $\mathrm{corr}[\mathcal{E}^{1,2},C_1/C_2]$
mean	0.028	0.149	0.445
(st.dev.)	(0.015)	(0.044)	(0.449)

Table 2: Statistics from 1000 simulations of the $N=2$ country economy. Each simulation has $T=100$ years and is designed as detailed in the caption of Figure 5.

6 Conclusion

This paper provides a theory of self-fulfilling fluctuations that are redistributive in nature. Theoretically, the existence of such self-fulfilling volatility relies on multiple markets and an endogenous force that connects asset valuations to some aspect of the real economy—our baseline model studies a growth-valuation link, but alternatives studied in the Internet Appendix include beliefs about growth rates (as in "price extrapolation" models), underinvestment wedges (as in "debt overhang" models), and entry/exit patterns (as in "creative destruction" models). Our framework helps explain the factor structure in firm-specific volatility and various dimensions of exchange rate disconnect such as the PPP puzzle and the Backus-Smith puzzle.

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Appendix

A Derivation of Equilibrium

In this appendix, we derive the complete set of equilibrium conditions that are used throughout the entire analysis. We write these conditions generally to accommodate both segmented and integrated financial markets.

Step 1: State prices. Each location has its own state-price density $\xi_{n,t}$, which follows

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \eta_t dB_t + \hat{\eta}_t \cdot d\hat{B}_t + \pi_{n,t} \cdot dZ_t\Big]. \tag{A.1}$$

The market prices of risk $(\eta, \hat{\eta})$ associated to $(B, \hat{B})$ are location-invariant, because markets for trading futures on these shocks are perfectly integrated. In the case of complete markets, the extrinsic shock risk prices are the same across locations, i.e., $\pi_{n,t} = \pi_t$ for each n. In the case of segmented equity markets, these risk prices may differ across locations.

In terms of these state prices, we have the no-arbitrage pricing relation for location-n equity:

$$\mu^q_{n,t} + g_{n,t} + \frac{1}{q_{n,t}} + \nu\varsigma^q_{n,t} + \hat{\nu}_{n,t} \cdot \hat{\varsigma}^q_{n,t} - r_t = (\nu + \varsigma^q_{n,t})\eta_t + (\hat{\nu}_{n,t} + \hat{\varsigma}^q_{n,t}) \cdot \hat{\eta}_t + \sigma^q_{n,t} \cdot \pi_{n,t}, \tag{A.2}$$

where with some abuse of notation we have defined the idiosyncratic risk exposure vector for $y_{n,t}$,

$$\hat{\nu}_{n,t} := \hat{\nu}\left[e_n - \begin{pmatrix} \alpha_{1,t} \\ \vdots \\ \alpha_{N,t} \end{pmatrix}\right] = \frac{1}{dt}\text{Cov}_t\Big[\frac{dy_{n,t}}{y_{n,t}}, d\hat{B}_t\Big], \tag{A.3}$$

where e_n is the nth elementary vector, and recall that $\alpha_{n,t} := y_{n,t}/Y_t$ are the endowment shares. Eq. (A.2) suffices to ensure no arbitrage in the equity market, so long as $q_{n,t} > 0$, which must hold in any equilibrium by free-disposal. The endowment share evolution is derived by applying Itô's formula to the definition of $\alpha_{n,t}$, namely

$$\frac{d\alpha_{n,t}}{\alpha_{n,t}} = (g_{n,t} - g_t)dt + \hat{\nu}_{n,t} \cdot d\hat{B}_t. \tag{A.4}$$

Step 2: Optimality. Integrating the dynamic budget constraint (3), using state-price dynamics (A.1), the pricing Eq. (A.2), and the individual transversality condition

$$\lim_{T\to\infty} \mathbb{E}_t[\xi_{n,T} w_{n,T}] = 0, \tag{A.5}$$

we obtain the standard static budget constraint

$$\mathbb{E}_t\Big[\int_t^\infty \frac{\xi_{n,s}}{\xi_{n,t}} c_{n,s} ds\Big] = w_{n,t}. \tag{A.6}$$

Note in passing that (A.6) implies $w_{n,t} > 0$, so the solvency constraint holds automatically. Agents' optimization problem is thus simply to maximize (5) subject to (A.6). The first-order condition of this optimization problem is

$$e^{-\delta t} c_{n,t}^{-\rho} = \xi_{n,t}. \tag{A.7}$$

Apply Itô's formula to Eq. (A.7) to obtain the following optimal consumption dynamics

$$\frac{dc_{n,t}}{c_{n,t}} = \frac{1}{\rho}\Big[r_t - \delta + \frac{\rho+1}{2\rho}\Big(\eta_t^2 + \|\hat{\eta}_t\|^2 + \|\pi_{n,t}\|^2\Big)\Big]dt + \frac{1}{\rho}\Big[\eta_t dB_t + \hat{\eta}_t \cdot d\hat{B}_t + \pi_{n,t}\cdot dZ_t\Big]. \tag{A.8}$$

To solve for the initial consumption $c_{n,t}$, given initial wealth $w_{n,t}$ and the dynamics of state prices and beliefs, substitute (A.7) back into (A.6) to get an equation for the wealth-consumption ratio

$$\omega_{n,t} := \frac{w_{n,t}}{c_{n,t}} = \mathbb{E}_t\Big[\int_t^\infty e^{-\delta(s-t)}\Big(\frac{c_{n,s}}{c_{n,t}}\Big)^{1-\rho} ds\Big]. \tag{A.9}$$

In general, Eq. (A.9) is useful because the dynamics of $c_{n,t}$ are given by Eq. (A.8) in terms of the state price density, so given all asset prices and initial wealth $w_{n,t}$, Eq. (A.9) allows us to compute $c_{n,t}$. (In particular, this will be useful when we study the log utility case with $\rho = 1$, since then Eq. (A.9) collapses to $w_{n,t}/c_{n,t} = \delta^{-1}$.) To instead represent (A.9) as a dynamic evolution equation, suppose

$$d\omega_{n,t} = \omega_{n,t}[\mu_{n,t}^{\omega}dt + \varsigma_{n,t}^{\omega}dB_t + \hat{\varsigma}_{n,t}^{\omega}\cdot d\hat{B}_t + \sigma_{n,t}^{\omega}\cdot dZ_t]$$

and then apply Itô's formula to $\xi_{n,t}\omega_{n,t}c_{n,t} = \mathbb{E}_t[\int_0^\infty \xi_{n,s}c_{n,s}ds] - \int_0^t \xi_{n,s}c_{n,s}ds$, and match drifts to obtain

$$\mu_{n,t}^{\omega} = \frac{\delta}{\rho} - \frac{1}{\omega_{n,t}} + \frac{\rho-1}{2\rho^2}\Big[\eta_t^2 + \|\hat{\eta}_t\|^2 + \|\pi_{n,t}\|^2\Big] + \frac{\rho-1}{\rho}\Big[r_t + \eta_t\varsigma_{n,t}^{\omega} + \hat{\eta}_t\cdot\hat{\varsigma}_{n,t}^{\omega} + \pi_{n,t}\cdot\sigma_{n,t}^{\omega}\Big]. \tag{A.10}$$

At the same time, since $\omega_{n,t} = w_{n,t}/c_{n,t}$, the wealth-consumption ratio diffusion coefficients are

$$\varsigma_{n,t}^{\omega} = \frac{\vartheta_{n,t}}{w_{n,t}} + \sum_{i=1}^{N}\frac{\theta_{n,i,t}}{w_{n,t}}(\nu + \varsigma_{i,t}^{q}) - \rho^{-1}\eta_t \tag{A.11}$$

$$\hat{\varsigma}_{n,t}^{\omega} = \frac{\hat{\vartheta}_{n,t}}{w_{n,t}} + \sum_{i=1}^{N}\frac{\theta_{n,i,t}}{w_{n,t}}(\hat{v}_{i,t} + \hat{\varsigma}_{i,t}^{q}) - \rho^{-1}\hat{\eta}_t \tag{A.12}$$

$$\sigma_{n,t}^{\omega} = \sum_{i=1}^{N}\frac{\theta_{n,i,t}}{w_{n,t}}\sigma_{i,t}^{q} - \rho^{-1}\pi_{n,t} \tag{A.13}$$

which identifies optimal portfolio choices $(\vartheta_n, \hat{\vartheta}_n)$, and partly identifies the equity holdings $(\theta_{n,i})$, given the wealth-consumption volatilities, asset price volatilities, and state price dynamics. With segmented equity markets, the equity holdings are also identified since $\theta_{n,i} = 0$ for $i \neq n$. Eqs. (A.11)-(A.13) simplify with log utility, since as mentioned earlier the wealth-consumption ratio is constant, $\omega_{n,t} = \delta^{-1}$. For instance, with $\rho = 1$ and segmented equity markets, Eq. (A.13) states that $\theta_{n,t}\sigma_{n,t}^{q} = w_{n,t}\pi_{n,t}$, so that $\sigma_{n,t}^{q}\cdot e_i > 0$ if and only if $\pi_{n,t}\cdot e_i > 0$, for each i.

Step 3: Aggregation. Recall the consumption shares $x_{n,t} := c_{n,t}/Y_t$. Using (A.8), apply Itô's formula to the goods market clearing condition $\sum_{n=1}^{N} c_{n,t} = Y_t$, and match drift and diffusion coefficients to obtain an equation for the riskless rate

$$r_t = \delta + \rho g_t - \frac{1}{2}\rho(\rho+1)\nu^2 - \frac{\rho+1}{2\rho}\sum_{n=1}^{N} x_{n,t}\|\pi_{n,t}\|^2 \tag{A.14}$$

expressions for the fundamental risk prices

$$\eta_t = \rho\nu \tag{A.15}$$

$$\hat{\eta}_t = 0 \tag{A.16}$$

and finally an equation linking the extrinsic risk prices

$$0 = \sum_{n=1}^{N} x_{n,t}\pi_{n,t}. \tag{A.17}$$

In the case that markets are complete, $\pi_{n,t} = \pi_t$ so Eq. (A.17) implies $\pi_t = 0$.

These expressions are all derived conditional on the consumption shares $(x_{n,t})_{n=1}^{N}$. Consumption share dynamics are obtained by applying Itô's formula to the definition of $x_{n,t}$, with the result being (after substituting several results above)

$$\frac{dx_{n,t}}{x_{n,t}} = \frac{\rho+1}{2\rho^2}\Big(\|\pi_{n,t}\|^2 - \sum_{i=1}^{N} x_{i,t}\|\pi_{i,t}\|^2\Big)dt + \frac{\pi_{n,t}}{\rho}\cdot dZ_t. \tag{A.18}$$

Next, the combination of bond and equity market clearing imply the aggregate wealth constraint

$$\sum_{n=1}^{N} w_{n,t} = \sum_{n=1}^{N} q_{n,t}y_{n,t}. \tag{A.19}$$

Apply equity and futures market clearing conditions to Eqs. (A.11)-(A.13), also using Eq. (A.19) and the expressions for the various risk prices, to obtain

$$\sum_{n=1}^{N} \alpha_{n,t}q_{n,t}\varsigma^{q}_{n,t} = \sum_{n=1}^{N} x_{n,t}\omega_{n,t}\varsigma^{\omega}_{n,t} \tag{A.20}$$

$$\sum_{n=1}^{N} \alpha_{n,t}q_{n,t}(\hat{v}_{n,t} + \hat{\varsigma}^{q}_{n,t}) = \sum_{n=1}^{N} x_{n,t}\omega_{n,t}\hat{\varsigma}^{\omega}_{n,t} \tag{A.21}$$

$$\sum_{n=1}^{N} \alpha_{n,t}q_{n,t}\sigma^{q}_{n,t} = \sum_{n=1}^{N} x_{n,t}\omega_{n,t}\Big[\rho^{-1}\pi_{n,t} + \sigma^{\omega}_{n,t}\Big] \tag{A.22}$$

In the case of segmented equity markets, Eq. (A.23) can be replaced by the stronger location-by-location condition

$$\alpha_{n,t}q_{n,t}\sigma^{q}_{n,t} = x_{n,t}\omega_{n,t}\Big[\rho^{-1}\pi_{n,t} + \sigma^{\omega}_{n,t}\Big] \tag{A.23}$$

Finally, let us also note the dynamics of the aggregate valuation ratio $Q_t := \sum_{n=1}^{N}\alpha_{n,t}q_{n,t}$, using Eqs. (A.2), (A.4), (A.15), and (A.16):

$$\begin{aligned} dQ_t = Q_t\Big[r_t - g_t + \rho\nu^2 - \frac{1}{Q_t} + (\rho-1)\nu\varsigma^{Q}_t + \sum_{n=1}^{N}\frac{\alpha_{n,t}q_{n,t}}{Q_t}\sigma^{q}_{n,t}\cdot\pi_{n,t}\Big]dt \\ + Q_t\Big[\varsigma^{Q}_t dB_t + \hat{\varsigma}^{Q}_t\cdot d\hat{B}_t + \sigma^{Q}_t\cdot dZ_t\Big], \end{aligned} \tag{A.24}$$

where the diffusions $(\varsigma^{Q}_t, \hat{\varsigma}^{Q}_t, \sigma^{Q}_t)$ are given by $\varsigma^{Q}_t := \sum_{n=1}^{N}\frac{\alpha_{n,t}q_{n,t}}{Q_t}\varsigma^{q}_{n,t}$ for the aggregate shock, $\hat{\varsigma}^{Q}_t := \sum_{n=1}^{N}\frac{\alpha_{n,t}q_{n,t}}{Q_t}(\hat{v}_{n,t} + \hat{\varsigma}^{q}_{n,t})$ for the idiosyncratic shocks, and $\sigma^{Q}_t := \sum_{n=1}^{N}\frac{\alpha_{n,t}q_{n,t}}{Q_t}\sigma^{q}_{n,t}$ for the extrinsic.

B Proofs

B.1 Proof of Theorem 1

First, let us compute the Jacobian J, by differentiating Eqs. (12) and (14) evaluated at the steady state $q_n = q^*$ for all n:

$$\left.\frac{\partial \dot{q}_{n,t}}{\partial q_{m,t}}\right|_{ss} = \begin{cases} \delta + (\rho-1)g - \lambda q^*, & m = n; \\ 0, & m \neq n. \end{cases}$$
$$\left.\frac{\partial \dot{q}_{n,t}}{\partial Q_t}\right|_{ss} = \rho\lambda q^*,$$

and

$$\left.\frac{\partial \dot{Q}_t}{\partial q_{m,t}}\right|_{ss} = 0, \quad \forall m$$
$$\left.\frac{\partial \dot{Q}_t}{\partial Q_t}\right|_{ss} = \delta + (\rho-1)g + \lambda(\rho-1)q^*$$

With these computations, we populate the entries of J.

Next, write out the equations of the eigenvalue problem $Jv = \eta v$:

$$(\delta + (\rho-1)g - \lambda q^*)v_n + \rho\lambda q^* v_{N+1} = \eta v_n, \quad 1 \leq n \leq N$$
$$(\delta + (\rho-1)g - \lambda q^* + \rho\lambda q^*)v_{N+1} = \eta v_{N+1}$$

If $v_n = v_{N+1}$ for all $n \leq N$, then the two equations become identical for any η. Since $v_{N+1} \neq 0$ in such case (otherwise the entire eigenvector would be zero), we obtain $\eta = \delta + (\rho-1)g + (\rho-1)\lambda q^*$. This corresponds to the single eigenvalue η_+ and its unique eigenvector $v(\eta_+) = \mathbf{1}_{N+1}$. If instead $v_n \neq v_{N+1}$ for any n, then we may take the difference between the two equations to obtain

$$(\delta + (\rho-1)g - \lambda q^*)(v_n - v_{N+1}) = \eta(v_n - v_{N+1}), \quad 1 \leq n \leq N,$$

which implies $\eta = \delta + (\rho-1)g - \lambda q^*$. In this case, it is clear that unless $\rho = 0$ or $\lambda = 0$ we must have $v_{N+1} = 0$. This corresponds to the eigenvalue η_-, which has multiplicity N because the set of vectors having $v_{N+1} = 0$ is N-dimensional. We can use the basis $(e_1, \ldots, e_N)$ for a basis of this N-dimensional space, hence our choice of the set of eigenvectors for $v(\eta_-)$.

Given the eigenvalues-eigenvectors, we want to prove that we can write

$$q_{n,t} = q^* + (q_{n,0} - Q_0)e^{\eta_- t} + (Q_0 - q^*)e^{\eta_+ t} + o\big(\|\mathbf{q}_0 - q^*\mathbf{1}_{N+1}\|\big), \quad n = 1, \ldots, N;$$
$$Q_t = q^* + (Q_0 - q^*)e^{\eta_+ t} + o\big(\|\mathbf{q}_0 - q^*\mathbf{1}_{N+1}\|\big),$$

where the terms $o(x)$ vanish faster than x as $x \to 0$, and where recall $\mathbf{q}_t := (q_{1,t}, \ldots, q_{N,t}, Q_t)'$. The algebra is as follows. First, note that J admits the eigen-decomposition $J = VDV^{-1}$, where

$$V = \begin{pmatrix} e_1 & e_2 & \cdots & e_N & \mathbf{1}_{N+1} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \eta_- & 0 & 0 & \cdots & 0 \\ 0 & \eta_- & 0 & \cdots & 0 \\ 0 & 0 & \eta_- & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta_+ \end{pmatrix}$$

Next, let $\mathbf{z}_t := \mathbf{q}_t - q^*\mathbf{1}_{N+1}$. Then, solving the linearly approximated differential equation $\dot{\mathbf{z}}_t \approx J\mathbf{z}_t$, we have

$$
\begin{aligned}
\mathbf{z}_t &\approx V\exp(Dt)V^{-1}\mathbf{z}_0 \\
&= \begin{pmatrix} e_1\exp(\eta_- t) & e_2\exp(\eta_- t) & \cdots & e_N\exp(\eta_- t) & ([\exp(\eta_+ t) - \exp(\eta_- t)]\mathbf{1}_N', e^{\eta_+ t})' \end{pmatrix}\mathbf{z}_0 \\
&= \begin{pmatrix} (z_{1,0} - z_{N+1,0})\exp(\eta_- t) + z_{N+1,0}\exp(\eta_+ t) \\ \vdots \\ (z_{N,0} - z_{N+1,0})\exp(\eta_- t) + z_{N+1,0}\exp(\eta_+ t) \\ z_{N+1,0}\exp(\eta_+ t) \end{pmatrix}
\end{aligned}
$$

To complete the proof of the theorem, we map the signs of the eigenvalues into the behavior of the valuations. If $\eta_+ > 0$, then Q_t necessarily deviates permanently from q^*. In such case, it is easy to show that the aggregate dynamics (14) feature a second steady state $q^{**} < q^*$, which is stable, but which is inconsistent with the location-specific dynamics (12). And so any equilibrium must feature $Q_t = q^*$ at all times.

On the other hand, if $\eta_+ < 0$, then the dynamics of Q_t are stable near q^*, meaning that any local deviation of Q_t from q^* eventually closes. In such case, there are a multiplicity of equilibria that may be indexed by Q_0, which may differ from q^*.

The analysis of the local prices is similar. If $\eta_- > 0$, then each local price has unstable dynamics, so $q_{n,t} = q^*$ at all times. If $\eta_- < 0$, then local prices have stable dynamics, so there exist a multiplicity of equilibria indexed by $(q_{n,0})_{n=1}^N$, with the restriction that $\sum_{n=1}^N \alpha_{n,0} q_{n,0} = Q_0$.

B.2 Existence and Uniqueness Theorem for BSDEs

Here, we cite a useful mathematical theorem from Briand and Confortola (2008) that helps us prove Lemma 1. We adapt their hypotheses and results to our situation with a finite-dimensional Brownian motion. In the results of this section, let B be a d-dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathscr{F}_t$ is the completion of the sigma-algebra generated by B.

Let τ be an $(\mathscr{F}_t)_{t\geq 0}$ stopping time, and let ξ be a bounded $\mathcal{F}_\tau$-measurable random variable. Consider the following backward stochastic differential equation (BSDE):

$$
dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t, \quad \text{where} \quad Y_\tau = \xi \text{ on } \{\tau < \infty\}, \tag{B.1}
$$

where the *generator* function f is a progressively-measurable mapping,

$$
f : \Omega \times [0, \infty) \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}
$$

and where $(y, z) \mapsto f(t, y, z)$ is continuous for all $t \geq 0$. A *solution* to the BSDE (B.1) is a pair of progressively measurable processes (Y, Z) such that Y is a path-continuous process; such that on $\{\tau < \infty\}$, we have $Y_t = \xi$ and $Z_t = 0$ for $t \geq \tau$; and such that $(f(t, Y_t, Z_t))_{t\in[0,T]}$ belongs to $\mathscr{L}^1(0, T; \mathbb{R})$ and $(Z_t)_{t\in[0,T]}$ belongs to $\mathscr{L}^2(0, T; \mathbb{R}^n)$ for every $T > 0$.

Suppose there exist two constants $\alpha > 0$ and $K \geq 0$ such that f satisfies the following hypotheses:

(H.i) $|f(t, y, z)| \leq K(1 + \|y\| + \|z\|^2)$ for all y, z

(H.ii) $|f(t, y, z) - f(t, y, z')| \leq K(1 + \|z\| + \|z'\|)\|z - z'\|$ for all y

(H.iii) $(y - y')(f(t, y, z) - f(t, y', z)) \leq -\alpha(y - y')^2$ for all y, y', z

A solution to the BSDE is a pair $(Y_t, Z_t)_{t\geq 0}$ of progressively-measurable processes such that (B.1) holds on every interval $[t, T]$. The following result is Theorem 3.3 in Briand and Confortola (2008).

Theorem B.1. *Under conditions (H.i)-(H.iii) above, there exists a unique solution (Y, Z) to the BSDE (B.1) such that Y is a bounded process.*

B.3 Proof of Lemma 1

Adding fundamental shocks. For the proof, we may generalize the equations listed in Section 3 to allow for aggregate and idiosyncratic fundamental shocks (i.e., $\nu > 0$ and $\hat{\nu} > 0$). The equations are the same as in Appendix A but where complete markets additionally imposes $\pi_{n,t}$ is independent of n. And so Eq. (A.17) implies $\pi_{n,t} = 0$ for all n.

Using the result $\pi_t = 0$ in Eq. (A.24), as well as the expression for r_t in (A.14) and the expression for growth g_t in (7), the aggregate valuation ratio Q_t satisfies

$$\begin{aligned} dQ_t = Q_t\Big[\delta + (\rho-1)(g-\lambda q^*) + (\rho-1)\lambda Q_t - \frac{1}{2}\rho(\rho-1)\nu^2 - \frac{1}{Q_t} + (\rho-1)\nu\varsigma_t^Q\Big]dt \\ + Q_t\Big[\varsigma_t^Q dB_t + \hat{\varsigma}_t^Q \cdot d\hat{B}_t + \sigma_t^Q \cdot dZ_t\Big], \end{aligned}$$

where $q^* := [\delta + (\rho-1)g - \frac{1}{2}\rho(\rho-1)\nu^2]^{-1}$ is now the deterministic steady state after accounting for the presence of aggregate shocks. As usual, we implicitly make parameter assumptions such that $q^* > 0$. (Note that there is a second value of Q_t that sets the drift above equal to zero, when $\varsigma_t^Q = 0$, but this value is negative, which is not possible in equilibrium with free disposal.)

Corner case: $\rho = 1$. If $\rho = 1$, then the formulas of Appendix A prove that each agent consumes δ fraction of her wealth, and so $Q_t = \delta^{-1} = q^*$ automatically by the aggregate wealth constraint. Therefore, the remainder of the proof assumes that $\rho > 1$.

Setting up the BSDE. The goal of the proof is to apply the existence/uniqueness Theorem B.1 in Section B.2. First, we write the problem in a way that fits the setting of Section B.2. Applying Itô's formula to $U_t := \log(Q_t/q^*)$, we have

$$\begin{aligned} dU_t = \Big[\delta + (\rho-1)(g-\lambda q^*) + (\rho-1)\lambda q^* \exp(U_t) - \frac{1}{2}\rho(\rho-1)\nu^2 - (q^*)^{-1}\exp(-U_t) + (\rho-1)\nu\varsigma_t^Q \\ - \frac{1}{2}(\varsigma_t^Q)^2 - \frac{1}{2}\|\hat{\varsigma}_t^Q\|^2 - \frac{1}{2}\|\sigma_t^Q\|^2\Big]dt + \varsigma_t^Q dB_t + \hat{\varsigma}_t^Q \cdot d\hat{B}_t + \sigma_t^Q \cdot dZ_t. \end{aligned}$$

Now, let us rewrite these in a more canonical form, by collecting all shocks and exposures into the Brownian shock vector $W := (B, \hat{B}', Z')'$ and the diffusion vector V_t:

$$dU_t = -f(U_t, V_t)dt + V_t dW_t \tag{B.2}$$

$$f(u, v) := \frac{1}{2}\|v\|^2 - (\rho-1)\nu v_1 - (\rho-1)\lambda q^*[\exp(u)-1] + (q^*)^{-1}[\exp(-u)-1], \tag{B.3}$$

where $v_1 := e_1 \cdot v$ is the first element of v. These dynamics constitute a 1-dimensional BSDE for (U, V). One solution to this BSDE is clearly $(U, V) = 0$ (i.e., $Q_t = q^*$ for all t).

Second, to be able to apply the results from Section B.2, despite the presence of the exponential function in (B.3), we need to "truncate and linearly extend" the generator f for extreme values of u.

In particular, let $L > 0$ be an arbitrary number. Define the linearly-extended generator f_L by

$$f_L(u,v) := \begin{cases} f(u,v), & \text{if } u \in [-L, L]; \\ f(u,v) + \Delta_L(u), & \text{if } u > L; \\ f(u,v) + \Delta_{-L}(u), & \text{if } u < -L. \end{cases} \tag{B.4}$$

where

$$\begin{aligned} \Delta_L(u) &:= -f(L,v) + (u-L)\frac{\partial f}{\partial u}(L,v) = (\rho-1)\lambda q^* \Big[\exp(u) - \exp(L) - \exp(L)(u-L)\Big] \\ &\qquad - (q^*)^{-1}[\exp(-u) - \exp(-L) + \exp(-L)(u-L)] \\ \Delta_{-L}(u) &:= -f(-L,v) + (u+L)\frac{\partial f}{\partial u}(-L,v) = (\rho-1)\lambda q^* \Big[\exp(u) - \exp(-L) - \exp(-L)(u+L)\Big] \\ &\qquad - (q^*)^{-1}[\exp(-u) - \exp(L) + \exp(L)(u+L)]. \end{aligned}$$

The linearly-extended generator $f_L(u,v)$ is continuous, as needed, and in fact is continuously differentiable.

The linearly-extended generator defines an alternative "linearly-extended BSDE"

$$dU_t = -f_L(U_t, V_t)dt + V_t dW_t. \tag{B.5}$$

Note that $(U,V) = 0$ is clearly a solution to the linearly-extended BSDE (B.5), for each $L > 0$. Our goal is to show that this solution is unique. Indeed, if we are able to prove this, then since L is arbitrary and can be made arbitrarily large, we will have proved that $(U,V) = 0$ is also the unique bounded solution to the original BSDE (B.2)-(B.3).

Verify the hypotheses of the BSDE theorem. We apply Theorem B.1 in Section B.2, with an almost-sure infinite stopping time ($\tau = +\infty$), in which case the "terminal condition" becomes irrelevant. We verify the assumptions (H.i)-(H.iii) directly preceding the theorem, which then proves that the solution $(U,V) = 0$ is the unique solution to (B.5).

Condition (H.i). By its linearly-extended construction, f_L has a maximum (absolute value) slope with respect to u of $K_u := \max[(\rho-1)\lambda q^* \exp(L), (q^*)^{-1}\exp(L)]$. Next, the (absolute value) slope of f_L with respect to $\|v\|^2$ is at most $K_v := \frac{1}{2} + |(\rho-1)\nu|$, which can be seen by applying the following basic inequality: $|(\rho-1)\nu v_1| \le |(\rho-1)\nu|\|v\|^2 + |(\rho-1)\nu|$. The remaining components of f_L that do not depend on (u,v) may be bounded by the constant $K_0 := |(\rho-1)\nu| + |(\rho-1)\lambda q^*|(1+\exp(L)L) + (q^*)^{-1}(1+\exp(L)L)$. Thus, hypothesis (H.i) of Section B.2 is satisfied with $K = \max[K_0, K_u, K_v]$.

Condition (H.ii). Second, we have

$$\begin{aligned} |f_L(u,v) - f_L(u,v')| &= \Big|\frac{1}{2}(\|v\|^2 - \|v'\|^2) - (\rho-1)\nu(v_1 - v_1')\Big| \\ &\le \frac{1}{2}\big|\|v\|^2 - \|v'\|^2\big| + \big|(\rho-1)\nu\big|\|v - v'\| \\ &\le \Big(\frac{1}{2}(\|v\| + \|v'\|) + \big|(\rho-1)\nu\big|\Big)\|v - v'\| \end{aligned}$$

where the third line uses the triangle inequality. Hence, hypothesis (H.ii) of Section B.2 holds with $K = \max[\frac{1}{2}, |(\rho-1)\nu|]$.

Condition (H.iii). Finally, to verify the strict monotonicity hypothesis, compute

$$\frac{\partial f_L(u,v)}{\partial u} = \begin{cases} -(\rho-1)\lambda q^* \exp(u) - (q^*)^{-1}\exp(-u), & \text{if } u \in [-L, L]; \\ -(\rho-1)\lambda q^* \exp(L) - (q^*)^{-1}\exp(-L), & \text{if } u > L; \\ -(\rho-1)\lambda q^* \exp(-L) - (q^*)^{-1}\exp(L), & \text{if } u < -L. \end{cases}$$

Note that, for any $L > 0$, we have $\alpha := \inf_{u\in[-L,L]}\{(\rho-1)\lambda q^* \exp(u) + (q^*)^{-1}\exp(-u)\} > 0$, because $\rho > 1$ and $\lambda \geq 0$. Consequently, we have

$$\frac{\partial f_L(u,v)}{\partial u} \leq -\alpha < 0,$$

which proves that hypothesis (H.iii) of Section B.2 holds.

Conclude. Having verified the hypotheses (H.i)-(H.iii), Theorem B.1 then implies the unique solution $(U,V) = 0$ is the unique one for the linearly-extended BSDE (B.5), for each $L > 0$. Since L can be made arbitrarily large, we then have that $(U,V) = 0$ is the unique solution, with U bounded, to the original BSDE (B.2).

B.4 Proof of Lemma 2

This proof follows a very similar procedure to that of Lemma 1. As stated, we assume that $\lambda < (\frac{1}{1+\varepsilon})^2(\frac{1}{1-\rho})(\frac{1}{q^*})^2$, with $\varepsilon > 0$ some arbitrary number.

The key modification is that we truncate and linearly-extend the generator (B.3) at different points. For any $L > 0$, and recalling $\varepsilon > 0$, define the linearly-extended generator $f_{L,\varepsilon}$ by

$$f_{L,\varepsilon}(u,v) := \begin{cases} f(u,v), & \text{if } u \in [-L, \log(1+\varepsilon)]; \\ f(u,v) + \Delta_\varepsilon(u), & \text{if } u > \log(1+\varepsilon); \\ f(u,v) + \Delta_L(u), & \text{if } u < -L. \end{cases} \tag{B.6}$$

where $f(u,v)$ is defined in (B.3) and

$$\begin{aligned} \Delta_\varepsilon(u) &:= (\rho-1)\lambda q^* \Big[\exp(u) - (1+\varepsilon) - (1+\varepsilon)(u - \log(1+\varepsilon))\Big] \\ &\quad - (q^*)^{-1}\Big[\exp(-u) - \frac{1}{1+\varepsilon} + \frac{1}{1+\varepsilon}(u - \log(1+\varepsilon))\Big] \\ \Delta_L(u) &:= (\rho-1)\lambda q^* \Big[\exp(u) - \exp(-L) - \exp(-L)(u+L)\Big] \\ &\quad - (q^*)^{-1}[\exp(-u) - \exp(L) + \exp(L)(u+L)] \end{aligned}$$

At this point, we can easily verify the hypotheses (H.i) and (H.ii) of Section B.2 in an identical manner to what we performed in the proof of Lemma 1. We omit this argument because it is identical. It remains to verify the monotonicity hypothesis (H.iii).

Compute

$$\frac{\partial f_{L,\varepsilon}(u,v)}{\partial u} = \begin{cases} -(\rho-1)\lambda q^* \exp(u) - (q^*)^{-1}\exp(-u), & \text{if } u \in [-L, \log(1+\varepsilon)]; \\ -(\rho-1)\lambda q^*(1+\varepsilon) - (q^*)^{-1}\frac{1}{1+\varepsilon}, & \text{if } u > \log(1+\varepsilon); \\ -(\rho-1)\lambda q^* \exp(-L) - (q^*)^{-1}\exp(L), & \text{if } u < -L. \end{cases}$$

It is easy to show that $\frac{\partial^2 f_{L,\varepsilon}(u,v)}{\partial u^2} > 0$ for $u \in [-L, \log(1+\varepsilon)]$. Therefore, the largest slope of this linearly-extended generator is

$$\sup_u \frac{\partial f_{L,\varepsilon}(u,v)}{\partial u} = -(\rho-1)\lambda q^*(1+\varepsilon) - (q^*)^{-1}\frac{1}{1+\varepsilon} < 0,$$

where the inequality uses the assumptions that $\rho < 1$ and $0 \leq \lambda < (\frac{1}{1+\varepsilon})^2(\frac{1}{1-\rho})(\frac{1}{q^*})^2$. In other words, we can set $\alpha := \sup_u \frac{\partial f_{L,\varepsilon}(u,v)}{\partial u} < 0$ in order to satisfy condition (H.iii) of Theorem B.2.

This proves that the solution $(U, V) = 0$ is the unique solution to the linearly-extended BSDE $dU_t = -f_{L,\varepsilon}(U_t, V_t)dt + V_t dW_t$. Because we can make L arbitrarily large in this argument, the same uniqueness point applies to the BSDE $dU_t = -f_{\infty,\varepsilon}(U_t, V_t)dt + V_t dW_t$, which only truncates and linearly extends for $u > \log(1+\varepsilon)$. Finally, because the lemma only requires us to consider solutions satisfying $Q_t \leq q^*(1+\varepsilon)$, i.e., $U_t \leq \log(1+\varepsilon)$, solving this latter linearly-extended BSDE suffices.

B.5 Proof of Lemma 3

It is easy to see directly that Lemma 3 constructs a redistributive set of diffusions. To see that *every* collection of redistributive diffusions that satisfies Assumption 1 can be constructed this way, refer back to Eq. (20), which recall is equivalent to Eq. (19). We may rewrite this equation as

$$\alpha_{n,t} q_{n,t} \sigma^q_{n,t} = \psi_t M e_n v^*_n,$$

where, due to Assumption 1, v^* is the unique vector in the null-space of M. After rearranging, we obtain Eq. (21). That every possible solution can be constructed follows from Step 1 of Lemma 3, which allows us to pick every possible v^* and corresponding matrix M. Finally, we also note that requiring $v^* \geq 0$ is without loss of generality, because the signs of any column of M can be flipped without changing its rank.

B.6 Proof of Proposition 1

The proposition only asks us to consider a redistributive equilibrium, so we have $Q_t = q^* = [\delta + (\rho-1)g - \frac{1}{2}\rho(\rho-1)\nu^2]^{-1}$ forever. The proof is almost identical to that of Proposition 2 below, so we provide a streamlined version.

First, using Eq. (A.8), Eq. (A.19), and $Q_t = q^*$ in Eq. (A.9), we must have constant wealth-consumption ratios $\omega_{n,t} = q^*$. Thus, we may set the loadings $\varsigma^q_{n,t}$ and $\hat{\varsigma}^q_{n,t}$ on dB_t and $d\hat{B}_t$, respectively, in an identical way to Proposition 2.

Using these results—along with $\pi_t = 0$, $\eta_t = \rho\nu$, $\hat{\eta}_t = 0$, $r_t = \delta\rho g - \frac{1}{2}\rho(\rho+1)\nu^2$, and $g_{n,t} = g + \lambda(q_{n,t} - q^*)$—in Eq. (A.2), we have

$$\begin{aligned}
&(\text{if } n \neq n^*_t) \quad dq_{n,t} = D(q_{n,t})dt + \frac{v^*_n}{\alpha_{n,t}} \psi_t (M e_n) \cdot dZ_t \\
&(\text{if } n = n^*_t) \quad dq_{n,t} = \Big[D(q_{n,t}) - \hat{v}_{n,t} \cdot \hat{\varsigma}^q_{n,t}\Big]dt + \frac{v^*_n}{\alpha_{n,t}} \psi_t (M e_n) \cdot dZ_t + q_{n,t} \hat{\varsigma}^q_{n,t} d\hat{B}_t
\end{aligned}$$

where

$$D(q) := -1 + \Big(\frac{1}{q^*} + \lambda q^*\Big) q - \lambda q^2$$

and where, if $\hat{\nu} \neq 0$ and $N \geq 3$, the location index n^*_t differs from $\arg\min_n q_{n,t}$ and $\arg\max_n q_{n,t}$. With this modification, the arguments go through identically, so we omit them here. In particular, conditions (P1)-(P2) allow us to verify that $D(\frac{\varepsilon+\lambda^{-1}}{q^*}) > 0$ and $D(Kq^*) < 0$, which allows us to prove that $(q_{n,t})_{n=1}^N$ are positive, bounded processes. Unlike Proposition 2, we do not need to examine the consumption shares $x_{n,t}$, because they are constant in the complete-markets case.

B.7 Proof of Proposition 2

Consider $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$ with $\lambda > \delta^2$ and fixed ϵ that satisfies $0 < \epsilon < \delta^{-2} - \lambda^{-1}$. Recall that $Q_t = q^* = \delta^{-1}$ holds in equilibrium. The general proof strategy is to conjecture asset price processes that feature extrinsic volatility and then verify that the conjectured dynamics are consistent with equilibrium—namely, asset valuations remain positive and bounded, and consumption shares remain positive.

Construction of SDE system. First, we have that $\alpha_{n,t}$ evolves via (A.4). Next, substitute the various equilibrium objects into (A.18) to obtain wealth distribution dynamics

$$dx_{n,t} = \psi_t^2 \delta^2 \Big[\frac{(v_n^*)^2}{x_{n,t}} - x_{n,t} \sum_{i=1}^N \frac{(v_i^*)^2}{x_{i,t}} \Big] dt + \psi_t \delta v_n^* (M e_n) \cdot dZ_t. \tag{B.7}$$

Finally, we construct the valuation dynamics. Follow Lemma 3 to construct $\sigma_{n,t}^q = \psi_t \frac{v_n^*}{\alpha_{n,t} q_{n,t}} M e_n$ for some matrix M with $\text{rank}(M) = N-1$, some v^* in the null-space of M, and some scalar process ψ_t. Given $\rho = 1$, Eq. (A.9) implies that all wealth-consumption ratios are constant over time and across locations at $\omega_{n,t} = \delta^{-1}$. Substituting $\omega_{n,t}$ and $\sigma_{n,t}^q$ into Eq. (A.23) implies

$$\pi_{n,t} = \frac{\delta v_n^* \psi_t}{x_{n,t}} M e_n.$$

We also conjecture an equilibrium with $\varsigma_{n,t}^q = 0$, which satisfies Eq. (A.20). Next, conjecture an equilibrium with the following idiosyncratic volatilities. If $\hat{v} \neq 0$ but $N \geq 3$, set $\hat{\varsigma}_{n,t}^q = 0$ for all $n \neq n_t^*$, where n_t^* is some location index that differs from $\arg\min_n q_{n,t}$ and $\arg\max_n q_{n,t}$ (such an index exists with probability one, if $N \geq 3$). If instead $\hat{v} = 0$, just set $\hat{\varsigma}_{n,t}^q = 0$ for all n, and let n_t^* be an arbitrary location index. To satisfy Eq. (A.21), we must then set

$$\hat{\varsigma}_{n_t^*,t}^q = -\sum_{n=1}^N \frac{q_{n,t}\alpha_{n,t}}{q_{n_t^*,t}\alpha_{n_t^*,t}} \hat{v}_{n,t}.$$

Substituting the above conjectures and all other equilibrium objects into the asset-pricing Eq. (A.2), we have

$$(\text{if } n \neq n_t^*) \quad dq_{n,t} = \Big[D(q_{n,t}) - \Big(\delta^2 \psi_t^2 \sum_{i=1}^N \frac{(v_i^*)^2}{x_{i,t}} \Big) q_{n,t} + \delta \frac{(v_n^* \psi_t)^2}{\alpha_{n,t} x_{n,t}} \Big] dt + \frac{v_n^*}{\alpha_{n,t}} \psi_t (M e_n) \cdot dZ_t \tag{B.8}$$

$$(\text{if } n = n_t^*) \quad dq_{n,t} = \Big[D(q_{n,t}) - \Big(\delta^2 \psi_t^2 \sum_{i=1}^N \frac{(v_i^*)^2}{x_{i,t}} \Big) q_{n,t} + \delta \frac{(v_n^* \psi_t)^2}{\alpha_{n,t} x_{n,t}} - \hat{v}_{n,t} \cdot \hat{\varsigma}_{n,t}^q \Big] dt + \frac{v_n^*}{\alpha_{n,t}} \psi_t (M e_n) \cdot dZ_t + q_{n,t} \hat{\varsigma}_{n,t}^q d\hat{B}_t \tag{B.9}$$

where

$$D(q) := -1 + (\delta + \lambda \delta^{-1}) q - \lambda q^2 \tag{B.10}$$

denotes the valuation drift when ψ vanishes. We have thus constructed an SDE system for $(\alpha_n, x_n, q_n)_{n=1}^N$, given the auxiliary volatility process ψ_t. Under this candidate construction, we use properties (P1) and (P2) in Proposition 2 to verify the conditions of equilibrium.

Boundedness of valuations. Define the domain

$$U := \Big\{ (\alpha_n, x_n, q_n)_{n=1}^N : \alpha_n > 0, \sum_{n=1}^N \alpha_n = 1, x_n > 0, \sum_{n=1}^N x_n = 1, q_n \in (\delta \lambda^{-1}, [K+1]\delta^{-1}) \Big\}$$

for $K > 1$ and let ∂U denote its boundary. Let $\tau := \{t \geq 0 : (\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N \notin U\}$ be the first exit time of the system from U. By inspection of Eqs. (A.4), (B.7), and (B.8)-(B.9), the assumed Property (P1), along with a suitable choice of the index n_t^*, ensures that all drift and diffusion coefficients of the SDE system are bounded. This is enough to ensure a unique strong solution to the SDEs exists up until time τ. We would like to prove that $\mathbb{P}[\tau = +\infty]$, so that we have a solution on the entire horizon that never leaves U.

To prove this, we will use Theorem 3.5, Remark 3.4, and Corollary 3.1 in Khasminskii (2011). Let $\mathscr{L}$ denote the infinitesimal generator of $\boldsymbol{u} := (\alpha_n, x_n, q_n)_{n=1}^N$. We are required to find a C^2 (Lyapunov) function v on U such that (i) $v \geq 0$ on U; (ii) $v \to +\infty$ uniformly as $\boldsymbol{u} \to \partial U$ and (iii) there exists some $c > 0$ such that $\mathscr{L}v \leq cv$ for all $\boldsymbol{u} \in U$ close enough to ∂U. [The fact that property (iii) is only required to hold near ∂U is due to Remark 3.4 in Khasminskii (2011), and this is also why we can exclude ψ from the SDE system, as properties (P1)-(P2) of the proposition require ψ to vanish as $\boldsymbol{x} \to \partial U$.]

We will use the Lyapunov function

$$v(\boldsymbol{u}) := \sum_{n=1}^N \Big[-\log(\alpha_n) - \log(x_n) + \frac{1}{q_n - \delta\lambda^{-1}} + \frac{1}{[K+1]\delta^{-1} - q_n} \Big].$$

First, $v \geq 0$ on U, so (i) holds (note that $\alpha_n < 1$ and $x_n < 1$ for all n on U, so the minus logarithm is always positive). Second, $v \to +\infty$ as $\boldsymbol{u} \to \partial U$ from the interior, so (ii) holds (check $\alpha_n \searrow 0$, $x_n \searrow 0$, $q_n \searrow \delta\lambda^{-1}$, or $q_n \nearrow [K+1]\delta^{-1}$). Third, compute

$$\begin{aligned}
\mathscr{L}v = &\sum_{n=1}^N \Big[-\lambda(q_n - \delta^{-1}) + \frac{1}{2}\|\nu_n\|^2 \Big] + \sum_{n=1}^N \Big[-\frac{1}{2}(\delta\psi)^2 (\frac{v_n^*}{x_n})^2 + \sum_{i=1}^N x_i (\frac{v_i^*}{x_i})^2 \Big] \\
&- \sum_{n=1}^N \Big(\frac{1}{(q_n - \delta\lambda^{-1})^2} - \frac{1}{([K+1]\delta^{-1} - q_n)^2} \Big) D(q_n) \\
&+ \sum_{n=1}^N \Big(\frac{1}{(q_n - \delta\lambda^{-1})^2} - \frac{1}{([K+1]\delta^{-1} - q_n)^2} \Big) \Big[(\delta\psi)^2 q_n \sum_{i=1}^N x_i (\frac{v_i^*}{x_i})^2 - \delta \frac{(\psi v_n^*)^2}{\alpha_n x_n} - \|\hat{v}_{n^*}\|^2 \frac{1}{N} \sum_{i=1}^N \frac{\alpha_i q_i}{\alpha_{n^*} q_{n^*}} \Big] \\
&+ \sum_{n=1}^N \Big(\frac{1}{(q_n - \delta\lambda^{-1})^3} + \frac{1}{([K+1]\delta^{-1} - q_n)^3} \Big) \Big[\psi^2 (\frac{v_n^*}{\alpha_n})^2 + \|\hat{v}_{n^*}\|^2 \frac{1}{N} \sum_{i=1}^N \frac{\alpha_i q_i}{\alpha_{n^*}} \Big]
\end{aligned} \tag{B.11}$$

The first summation (involving dynamics of α_n) is always bounded on U. The second summation (involving dynamics of x_n) converges to $-\infty$ as $x_n \to 0$ for any n. The remaining terms involve the dynamics of q_n. By suitable choice of the index process n_t^* (relevant only if $\hat{v} > 0$), we may always ensure that the terms above involving n^* are bounded. Then, note that for q_n sufficiently close to either $\delta\lambda^{-1}$ or $[K+1]\delta^{-1}$, $\psi = 0$ as required by condition (P2) of the proposition. Thus, as q_n approaches either of these boundaries, the only relevant term becomes the third summation. By inspection, $D(q) > 0$ if and only if $q \in (\delta\lambda^{-1}, \delta^{-1})$, so this summation approaches $-\infty$ as $q_n \searrow \delta\lambda^{-1}$ or $q_n \nearrow [K+1]\delta^{-1}$. Finally, the corner cases (i.e., multiple states approaching the boundary ∂U simultaneously) are handled as linear combinations of this analysis. Putting these pieces together, we have proven (iii), that $\mathscr{L}v \leq cv$ for some $c > 0$ and for $\boldsymbol{u}$ near enough to ∂U.

This verifies all the assumptions of Remark 3.4 and Corollary 3.1 in Khasminskii (2011), which proves that $\mathbb{P}[\tau = +\infty]$. In particular,

$$\mathbb{P}\Big\{ q_{n,t} \in \big(\delta\lambda^{-1}, K\delta^{-1}\big] \ \forall t \Big\} = 1,$$

which means that $\{(q_{n,t})_{n=1}^N : t \geq 0\}$ is positive and bounded almost-surely. Note that free disposal automatically holds by the positivity of these valuations.

Survival of consumption shares. Next, we show that $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} x_{n,T}^{-1}] = 0$. Apply Itô's formula to x_n, using dynamics (B.7), to write

$$e^{-\delta T}\frac{1}{x_{n,T}} = \frac{G_{n,T}}{x_{n,0}} \exp\Big[\delta \int_0^T \Big(\delta\psi_t^2 \sum_{i=1}^N \frac{1}{x_{i,t}}(v_i^*)^2 - 1\Big)dt\Big],$$

where G_n is the martingale defined by

$$G_{n,t} = \exp\Big[-\frac{1}{2}\int_0^t (\delta\psi_s)^2 (\frac{v_n^*}{x_{n,s}})^2 ds - \int_0^t (\delta\psi_s)(\frac{v_n^*}{x_{n,s}})^2 (Me_n)\cdot dZ_s\Big].$$

So using G_n as a change-of-measure to define the expectation $\tilde{\mathbb{E}}^n$, we can compute

$$\mathbb{E}_0[e^{-\delta T}(x_{n,T})^{-1}] = \frac{1}{x_{n,0}}\tilde{\mathbb{E}}_0^n \exp\Big[\delta \int_0^T \Big(\delta\psi_t^2 \sum_{i=1}^N \frac{1}{x_{i,t}}(v_i^*)^2 - 1\Big)dt\Big]$$

A sufficient condition for this to vanish as $T\to\infty$ is $\delta\psi_t^2 \sum_{i=1}^N \frac{1}{x_{i,t}}(v_i^*)^2 < 1$ for every t. But indeed we have

$$\delta\psi_t^2 \sum_{i=1}^N \frac{1}{x_{i,t}}(v_i^*)^2 < \frac{\delta\psi_t^2}{\min_n x_{n,t}} \sum_{i=1}^N (v_i^*)^2 = \frac{\delta\psi_t^2}{\min_n x_{n,t}} < 1.$$

The first inequality uses the definition of minimum; the equality uses $\|v^*\| = 1$; and the last inequality uses requirement (P1). This proves the result.

Verify No-Bubble and No-Ponzi conditions. At this point, it remains to verify that the No-Ponzi conditions hold. We actually start by verifying the no-bubble Condition 1:

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}_t[\xi_{n,T} q_{n,T} y_{n,T}] &= \lim_{T\to\infty} \mathbb{E}_t[\alpha_{n,T} q_{n,T} e^{-\delta T}\frac{1}{x_{n,T}}] \\ &\le \lim_{T\to\infty} \mathbb{E}_t[q_{n,T} e^{-\delta T}\frac{1}{x_{n,T}}] \\ &\le K\delta^{-1} \lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T}\frac{1}{x_{n,T}}] = 0.\end{aligned}$$

In the first line, we have used (A.7); in the second line, we have used the fact that $\alpha_{n,T} \le 1$; in the third line, we have used the boundedness of q_n by $K\delta^{-1}$, and then the result proved earlier that $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T}\frac{1}{x_{n,T}}] = 0$. This proves that Condition 1 holds.

Next, note that $w_{n,t} = \delta^{-1} c_{n,t} = \delta^{-1} x_{n,t} Y_t$, so that $w_{n,t} \ge 0$ if and only if $x_{n,t} \ge 0$. The latter inequality is proved by inspecting the dynamics (A.18).

Now, since $w_{n,t}$ and $q_{n,t}$ are both positive, and since $\xi_{n,t}$ is the local state-price density, we know $(\xi_{n,t} w_{n,t})_{t\ge 0}$ and $(\xi_{n,t} y_{n,t} q_{n,t})_{t\ge 0}$ are both continuous, positive super-martingales. So by Doob's super-martingale convergence theorem, we know that $\lim_{T\to\infty} \xi_{n,T} w_{n,T}$ and $\lim_{T\to\infty} \xi_{n,T} y_{n,T} q_{n,T}$ both exist and are finite. Next, transversality condition (A.5) and no-bubble Condition 1 imply there exists a sub-sequence of times $\{T_j\}_{j=1}^\infty$ along which $\lim_{j\to\infty} \xi_{n,T_j} w_{n,T_j} = 0$ and $\lim_{j\to\infty} \xi_{n,T_j} y_{n,T_j} q_{n,T_j} = 0$. But these limits must be the same along any subsequence, by the first step (i.e., that the limits exist), which shows $\lim_{T\to\infty} \xi_{n,T} w_{n,T} = \lim_{T\to\infty} \xi_{n,T} y_{n,T} q_{n,T} = 0$. Finally, combine the previous limits with equity market clearing $\theta_{n,T} = q_{n,T} y_{n,T}$ to obtain (4).

Internet Appendix
(Not for publication)

Segmentation and Beliefs:
A Theory of Self-Fulfilling Idiosyncratic Risk

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C Other stabilizing forces

This online appendix provides three additional microfoundations for sources of endogeneity that keep valuation ratios stable—therefore, we call these *stabilizing forces*. In Section C.1, we replace the growth-valuation link with a connection between valuations and beliefs about growth. In Section C.2, we model firms that invest, subject to a debt overhang problem, which microfounds connection between valuations and growth—this is similar to our baseline model but with a particular microfoundation. In Section C.3, we model a creative destruction process that depends on valuations. In all of the extensions in this appendix, we assume that agents have log utility ($\rho = 1$).

C.1 Valuation-dependent beliefs as a "stabilizing force"

In the main text, we study a positive connection between asset valuations and growth. Here, we explore a model in which asset valuations increase *beliefs about growth* rather than actual growth. For reasons that will become clear, self-fulfilling volatility requires segmented futures markets (i.e., no cross-location trading on the dB_t shock); if futures markets were integrated, all agents would agree on the aggregate risk price, and beliefs would not affect asset valuations. Unfortunately, the analysis of this setting is substantially more complex than our baseline model, so we specialize to an economy with constant true growth rates g, without any idiosyncratic risk ($\hat{\nu} = 0$), and with an additional cross-location entry/exit margin that facilitates analysis of the wealth distribution. More details on this entry/exit margin below. Furthermore, we eventually specialize to a two-location economy, in which one location is vanishingly small (like a small open economy).

Endowments. Each location receives identical geometric Brownian motions

$$\frac{dy_{n,t}}{y_{n,t}} = gdt + \nu dB_t$$

Therefore, the aggregate output also follows $dY_t/Y_t = gdt + \nu dB_t$. Furthermore, each locations' endowment share is constant over time. Therefore, we write α_n for the location-n endowment share, dropping the time subscript.

Beliefs. Let $\mathbb{P}$ be the objective probability measure. Subjective beliefs are modeled as follows. For some process $\gamma_{n,t}$, we define the likelihood ratio between subjective beliefs and the physical probability as

$$H_{n,t} := \Big(\frac{d\widetilde{\mathbb{P}}^n}{d\mathbb{P}}\Big)_t = \exp\Big[\int_0^t \gamma_{n,s} dB_s - \frac{1}{2}\int_0^t \gamma_{n,s}^2 ds\Big]. \tag{C.1}$$

By Girsanov's theorem, this amounts to assuming that agents in location n believe that $d\tilde{B}_{n,t} := dB_t - \gamma_{n,t}dt$ is a Brownian motion. Meanwhile, agents have rational beliefs about all other shocks. As with the endogeneity in fundamental growth rates in Eq. (6), we assume that

$$\gamma_{n,t} = \frac{\lambda}{\nu}(q_{n,t} - \delta^{-1}), \quad \lambda > 0. \tag{C.2}$$

Equation (C.2) says that investors become more optimistic about growth when prices rise. An implication of these assumptions is that agent n holds the following subjective belief $\tilde{g}_{n,t} := \frac{1}{dt}\tilde{\mathbb{E}}^n_t[\frac{dy_{n,t}}{y_{n,t}}]$ about his local endowment growth rate:

$$\tilde{g}_{n,t} = g + \lambda(q_{n,t} - \delta^{-1}). \tag{C.3}$$

This mirrors Eq. (6), but for perceived growth rather than true growth.

Valuations. In general, as there are no $d\hat{B}_t$ shocks, asset valuations take the form

$$\frac{dq_{n,t}}{q_{n,t}} = \mu^q_{n,t}dt + \varsigma^q_{n,t}dB_t + \sigma^q_{n,t} \cdot dZ_t$$

However, we conjecture an equilibrium in which $\varsigma^q_{n,t} = 0$ for all n.

Optimization and risk prices. Without hedging markets for the aggregate dB_t shock, location n has its own aggregate risk price, and its SDF follows

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \eta_{n,t}dB_t + \pi_{n,t} \cdot dZ_t\Big].$$

Different to the baseline model, marginal utility incorporates the belief distortion, so optimal consumption sets

$$H_{n,t}e^{-\delta t}c^{-1}_{n,t} = \xi_{n,t}.$$

Thus, optimal consumption dynamics for each location n are then

$$\frac{dc_{n,t}}{c_{n,t}} = \Big[r_t - \delta - \gamma_{n,t}(\gamma_{n,t} + \eta_{n,t}) + (\gamma_{n,t} + \eta_{n,t})^2 + \|\pi_{n,t}\|^2\Big]dt + (\gamma_{n,t} + \eta_{n,t})dB_t + \pi_{n,t} \cdot dZ_t. \tag{C.4}$$

As before, with log utility, the location-n wealth-consumption ratio is equal to $\omega_{n,t} := \frac{w_{n,t}}{c_{n,t}} = \delta^{-1}$. Apply Itô's formula to this result, using the dynamic budget constraint (3) with the following substitutions: $\vartheta_{n,t} = 0$ (since there are no futures markets), η_t replaced by the location-specific risk price $\eta_{n,t}$ (again, since there are no futures markets), $\theta_{n,t} = q_{n,t}y_{n,t}$ (equity market clearing), and imposing the conjecture $\varsigma^q_{n,t} = 0$. The results are

$$\eta_{n,t} + \gamma_{n,t} = \frac{\delta\alpha_n q_{n,t}}{x_{n,t}}\nu \tag{C.5}$$

$$\pi_{n,t} = \frac{\delta\alpha_n q_{n,t}}{x_{n,t}}\sigma^q_{n,t}. \tag{C.6}$$

In other words, the risk exposures of representative agent n coincide with the risks they hold through their local equity.

Aggregation. Applying Itô's formula to the goods market clearing condition $\sum_{n=1}^{N} c_{n,t} = Y_t$, we obtain

$$r_t = \delta + g + \sum_{n=1}^{N} x_{n,t}\gamma_{n,t}(\gamma_{n,t} + \eta_{n,t}) - \sum_{n=1}^{N} x_{n,t}[(\gamma_{n,t} + \eta_{n,t})^2 + \|\pi_{n,t}\|^2] \tag{C.7}$$

from matching drifts, and

$$\sum_{n=1}^{N} \alpha_n q_{n,t} = \delta^{-1} \tag{C.8}$$

$$\sum_{n=1}^{N} \alpha_n q_{n,t}\sigma_{n,t}^{q} = 0 \tag{C.9}$$

from matching diffusion coefficients and substituting Eqs. (C.5)-(C.6) above for $\eta_{n,t}$ and $\pi_{n,t}$. Eq. (C.8) is simply the aggregate wealth constraint. Eq. (C.9) is a constraint on the relative extrinsic volatilities. To satisfy this constraint, follow Step 1 of Lemma 3 to pick a matrix M and vector v^*. Then, introduce a positive process ψ_t (as in Proposition 2) and let

$$\alpha_n q_{n,t}\sigma_{n,t}^{q} = \psi_t v_n^* M e_n. \tag{C.10}$$

Clearly, Eq. (C.10) solves Eq. (C.9). The dynamics of $x_{n,t} = c_{n,t}/Y_t$ are given by applying Itô's formula to its definition:

$$\begin{aligned}\frac{dx_{n,t}}{x_{n,t}} = \Big[r_t - \delta - g - \gamma_{n,t}(\gamma_{n,t} + \eta_{n,t}) - \nu(\gamma_{n,t} + \eta_{n,t}) + \nu^2 + (\gamma_{n,t} + \eta_{n,t})^2 + \|\pi_{n,t}\|^2\Big]dt \\ + (\gamma_{n,t} + \eta_{n,t} - \nu)dB_t + \pi_{n,t} \cdot dZ_t.\end{aligned} \tag{C.11}$$

Finally, the equilibrium asset-pricing condition is

$$\mu_{n,t}^{q} + g + \frac{1}{q_{n,t}} - r_t = \nu\eta_{n,t} + \sigma_{n,t}^{q} \cdot \pi_{n,t}. \tag{C.12}$$

This completes the set of equilibrium equations, analogous to Appendix A. The key question is whether the dynamics above induce stationary valuations $(q_{n,t})_{n=1}^{N}$ and stationary consumption shares $(x_{n,t})_{n=1}^{N}$.

Entry/exit margin. We assume in reduced-form that entry/exit occurs between the locations in a way that keeps $\eta_{n,t} + \gamma_{n,t} \leq \bar{\eta}$ for all n, t. Such an assumption is reasonable, because the Sharpe ratios represent risk-adjusted profits to investors. In fact, with log utility, with an entry cost that is proportional to wealth, and in an equilibrium without self-fulfilling volatility, this is actually the optimal entry process, as shown in Khorrami (2022). Different entry costs map into different values of $\bar{\eta}$. We assume $\bar{\eta} > \nu$, i.e., entry occurs when Sharpe ratios are somewhat above the perfect risk-sharing Sharpe ratio. Using Eq. (C.5), such an entry process translates into a lower bound for $x_{n,t}$:

$$x_{n,t} \geq \underline{x}_{n,t} := \bar{\eta}^{-1}\delta\alpha_n q_{n,t}\nu. \tag{C.13}$$

When $x_{n,t}$ falls, Sharpe ratios rise, which provides an incentive for investors to flow from other locations into location n, keeping $x_{n,t} \geq \underline{x}_{n,t}$. Thus, $\underline{x}_{n,t}$ is a reflecting boundary for $x_{n,t}$. Modeling entry in this way substantially simplifies the analysis of the equilibrium dynamical system.

Steady state. The equilibrium dynamical system for $(x_{n,t}, q_{n,t})_{n=1}^N$ is governed by Eqs. (C.12) and (C.11). If there is no self-fulfilling volatility, $\psi_t = 0$, then this dynamical system has a deterministic steady state which is given by $x_{n,t} = \alpha_n$ and $q_{n,t} = \delta^{-1}$ for all n. Although the stability properties of the dynamical system are much more complicated in this model than in our baseline model, by specializing to $N = 2$ locations and treating one location as "small", we may obtain some sharp analytical results.

Example with one small and one large location. To transparently establish the existence of a sunspot equilibrium, we now specialize to $N = 2$ locations. With $N = 2$, we can focus on location-1 and determine the location-2 equilibrium objects via market clearing. In particular, drop the location subscripts and denote $\alpha := \alpha_1$, $x_t := x_{1,t}$, and $q_t := q_{1,t}$. Then, the location-2 objects are $\alpha_2 = 1 - \alpha$, $x_{2,t} = 1 - x_t$, and

$$q_{2,t} = \frac{\delta^{-1} - \alpha q_t}{1 - \alpha}$$

Furthermore, we assume that

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{so that} \quad v^* = (1, 1)' \in \text{null}(M).$$

This specification is equivalent to assuming there is only one extrinsic shock. Therefore, let us define abuse notation and define $Z_t := Z_{1,t}$.

Let us focus now on the location-1 valuation q_t and consumption share x_t. Substitute Eqs. (C.2), (C.5), (C.6), and (C.10) into Eq. (C.12) to obtain

$$dq_t = \Big[-1 + \frac{\delta \psi_t^2}{\alpha x_t} + \Big(r_t - g + \lambda \delta^{-1} \Big) q_t - \Big(\lambda - \frac{\delta \alpha}{x_t} \nu^2 \Big) q_t^2 \Big] dt + \frac{\psi_t}{\alpha} dZ_t. \tag{C.14}$$

Then, substituting (C.7) into (C.14) and doing some algebra, we obtain

$$dq_t = \Big[-1 + \Big(\frac{\delta}{\alpha x_t} - \frac{\delta^2 q_t}{x_t(1 - x_t)} \Big) \psi_t^2 + A_{1,t} q_t + A_{2,t} q_t^2 + A_{3,t} q_t^3 \Big] dt + \frac{\psi_t}{\alpha} dZ_t \tag{C.15}$$

$$\begin{aligned} \text{where} \quad A_{1,t} &:= \delta + \frac{\lambda \delta^{-1}}{1 - \alpha} - \frac{\nu^2}{1 - x_t} \\ A_{2,t} &:= \alpha \Big(\frac{\delta \nu^2}{x_t} + \frac{2 \delta \nu^2}{1 - x_t} - \frac{2\lambda}{1 - \alpha} \Big) - \lambda \\ A_{3,t} &:= \alpha \Big(\frac{\lambda \delta}{1 - \alpha} - \frac{\alpha (\delta \nu)^2}{x_t (1 - x_t)} \Big) \end{aligned}$$

Similarly, substitute various results into Eq. (C.11), we obtain

$$\begin{aligned} dx_t = \Big[& x_t \frac{\lambda \delta^{-1} \alpha}{1 - \alpha} (1 - \delta q_t)^2 - \lambda (q_t - \delta^{-1}) \alpha \delta q_t - \alpha \delta q_t \nu^2 \\ & + \frac{(\alpha \delta q_t)^2}{x_t} \nu^2 - \frac{(x_t - \alpha \delta q_t)^2}{1 - x_t} \nu^2 + \frac{(\delta \psi_t)^2}{x_t} - \frac{(\delta \psi_t)^2}{1 - x_t} \Big] dt + (\delta \alpha q_t - x_t) \nu dB_t + \delta \psi_t dZ_t. \end{aligned} \tag{C.16}$$

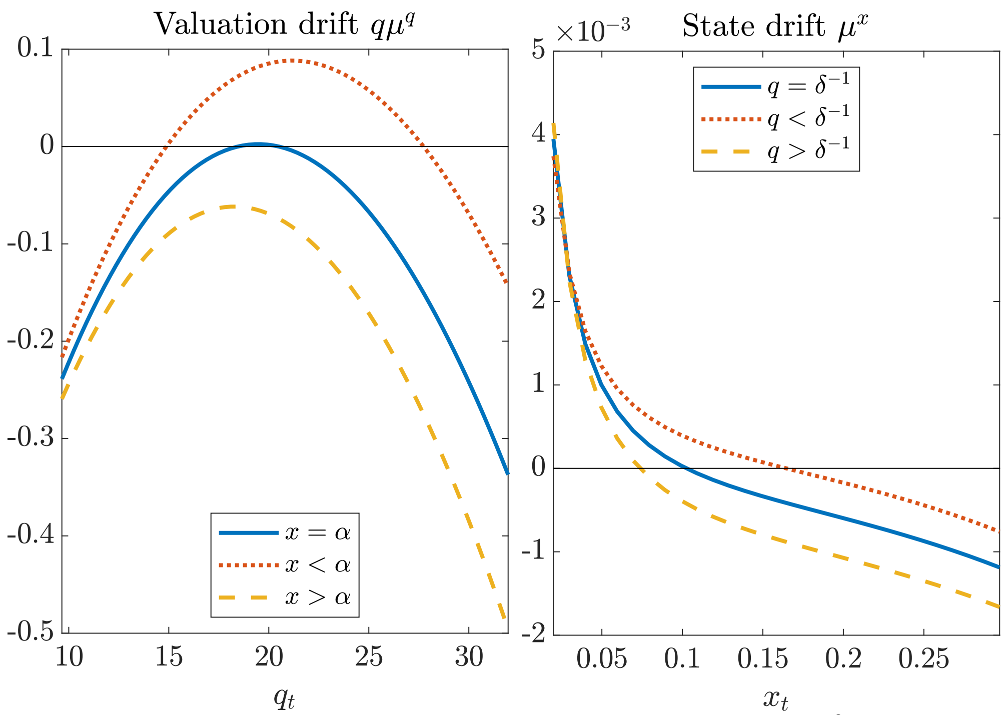
Given the entry process, consumption shares also obey $x_t \geq \bar{\eta}^{-1} \delta \nu \alpha q_t$ and $1 - x_t \geq \bar{\eta}^{-1} \delta \nu (1 - \alpha) q_{2,t}$. Combining these bounds and using the expression for $q_{2,t}$, equilibrium has

$$\bar{\eta}^{-1} \delta \nu \alpha q_t \leq x_t \leq 1 + \bar{\eta}^{-1} \delta \nu \alpha q_t - \bar{\eta}^{-1} \nu. \tag{C.17}$$

Equilibrium requires the dynamics (C.15) to be such that $q_t > 0$ and $q_t < \delta^{-1}/\alpha$ (so that $q_{2,t} > 0$) for all t.

Figure C.1 provides an illustration of the drifts of Eqs. (C.15) and (C.16) when $\psi_t = 0$. The dynamics look like they could be locally stable (see the solid and dotted lines in the left panel, near the higher steady state), but this conclusion seems to depend on the level of x_t relative to α (consumption versus endowment shares). Of course, this figure also depends on a specific choice of other parameters. So the question is whether some more general statements can be made about dynamical stability.

Figure C.1: Valuation and consumption share dynamics.



Notes. Parameters are $\delta = 0.05$, $g = 0.02$, $\nu = 0.1$, $\alpha = 0.1$, $\lambda = \frac{\delta^2}{1-\delta} + \delta\nu^2$.

Proving the general stationarity of $(q_t)_{t\geq 0}$ is technically difficult, so we sketch the main ideas in a limiting case in which one location is vanishingly small. This is essentially a "small open economy" limit. In particular, for each α, the equilibrium is indexed as follows. Let $\psi_t = \alpha\psi_t^*$ be the self-fulfilling volatility process (this intentionally vanishes with α). Let x_t^α and q_t^α be the resulting consumption share and valuation in equilibrium. Thus, $(x_t, q_t, \psi_t)_{t\geq 0} = (x_t^\alpha, q_t^\alpha, \alpha\psi_t^*)_{t\geq 0}$ is the equilibrium for a fixed α. We take $\alpha \to 0$ and establish the desired stability properties in that limiting equilibrium. Let $x_t^* := \lim_{\alpha\to 0} x_t^\alpha$ and $q_t^* := \lim_{\alpha\to 0} q_t^\alpha$ be the limiting equilibrium objects.

In this limiting equilibrium, $x_t^* = 0$ with probability 1. Indeed, inspecting the dynamics (C.16) with $\alpha \to 0$ and $\psi_t = \alpha\psi_t^* \to 0$, we see that

$$dx_t^* = -(x_t^*\nu)^2 dt - x_t^*\nu dB_t.$$

The initial consumption share of location 1 is $x_0^\alpha = \alpha\delta q_0^\alpha$, so $x_0^* = 0$. Using the dynamics above, we then have $x_t^* = 0$ for all t.

Define $\tilde{x}_t^* := \lim_{\alpha \to 0} x_t^\alpha / \alpha$, and note its initial value $\tilde{x}_0^* = \delta q_0^*$. Given the entry/exit margin, captured in Eq. (C.13), we have $\tilde{x}_t^* \geq \bar{\eta}^{-1} \delta \nu q_t^*$ (the upper bound scaled by $1/\alpha$ diverges and becomes irrelevant as α shrinks).

We can now examine the limiting dynamics for q_t^* and $\tilde{x}_t^*$:

$$dq_t^* = \Big[-1 + \frac{\delta}{\tilde{x}_t^*}(\psi_t^*)^2 + \Big(\delta + \lambda\delta^{-1} - \nu^2\Big)q_t^* + \Big(\frac{\delta\nu^2}{\tilde{x}_t^*} - \lambda\Big)(q_t^*)^2\Big]dt + \psi_t^* dZ_t \tag{C.18}$$

$$d\tilde{x}_t^* = \Big[\frac{(\delta q_t^*)^2}{\tilde{x}_t^*}\nu^2 - \delta q_t^*[\nu^2 + \lambda(q_t^* - \delta^{-1})] + \frac{(\delta\psi_t^*)^2}{\tilde{x}_t^*}\Big]dt + (\delta q_t^* - \tilde{x}_t^*)\nu dB_t + \delta\psi_t^* dZ_t \tag{C.19}$$

$$\text{where} \quad \tilde{x}_t^* \geq \bar{\eta}^{-1}\delta\nu q_t^*. \tag{C.20}$$

A steady state of this system is $(\tilde{x}_t^*, q_t^*, \psi_t^*) = (1, \delta^{-1}, 0)$. To show that self-fulfilling volatility is possible (i.e., $\psi_t^* \neq 0$), we need to show that $q_t^* > 0$ for all t with probability 1. To do this, we need the following parameter restrictions:

$$\delta > \nu^2 \tag{C.21}$$

$$\lambda > \delta^2 + \delta\nu^2 + 2\nu\delta^{1.5} \tag{C.22}$$

$$\nu < \bar{\eta} < \frac{1}{2}\frac{\delta + \lambda\delta^{-1} - \nu^2}{\nu} \tag{C.23}$$

Note that (C.21)-(C.23) are mutually consistent (i.e., the proposed interval for $\bar{\eta}$ is non-empty).

Consider the first-passage time

$$\tau := \Big\{t \geq 0 : q_t^* \leq \underline{q}_t^* := \frac{\delta + \lambda\delta^{-1} - \nu^2}{2(\lambda - \delta\nu^2/\tilde{x}_t^*)}\Big\}. \tag{C.24}$$

Let $\psi_\tau^* = 0$, so that self-fulfilling volatility vanishes as valuations reach the lower bound specified in (C.24). Then, we have the following lemma, which shows that the equilibrium is stable and therefore permits self-fulfilling volatility.

Lemma C.1. *Under parameter assumptions (C.21)-(C.23), we have $dq_\tau^* > 0$ almost-surely, and consequently $(q_t^*)_{t\geq 0} > 0$ given any process $(\psi_t^*)_{t\geq 0}$ that vanishes as q_t^* approaches $\underline{q}_t^*$.*

Proof. We first conjecture and then verify that $\tilde{x}_t^* > \delta\nu^2/\lambda$. Given this conjecture, notice from the definition of $\underline{q}_t^*$ in (C.24) that

$$\underline{q}_t^* > \frac{\delta + \lambda\delta^{-1} - \nu^2}{2\lambda}. \tag{C.25}$$

Under parameter assumption (C.21), the right-hand-side of the expression above is strictly positive. Combine parameter assumption (C.23) with the entry barrier in (C.20), along with $q_t^* \geq \underline{q}_t^*$ and the lower bound for $\underline{q}_t^*$ in (C.25). The result is that we verify

$$\tilde{x}_t^* > \frac{\delta\nu^2}{\lambda}. \tag{C.26}$$

Next, we need to show that $dq_\tau^* > 0$ if $\psi_\tau^* = 0$. Consider the function

$$f(q; x) := -1 + \Big(\delta + \lambda\delta^{-1} - \nu^2\Big)q + \Big(\frac{\delta\nu^2}{x} - \lambda\Big)q^2$$

Note that $dq^*_\tau = f(q^*_\tau; \tilde{x}^*_\tau)dt$. As a function of q, $f(q;x)$ is a quadratic function with two roots q_+ and q_-, which are

$$q_+(x) = \frac{\delta + \lambda\delta^{-1} - \nu^2 + \sqrt{(\delta + \lambda\delta^{-1} - \nu^2)^2 - 4(\lambda - \frac{\delta\nu^2}{x})}}{2(\lambda - \frac{\delta\nu^2}{x})}$$

$$q_-(x) = \frac{\delta + \lambda\delta^{-1} - \nu^2 - \sqrt{(\delta + \lambda\delta^{-1} - \nu^2)^2 - 4(\lambda - \frac{\delta\nu^2}{x})}}{2(\lambda - \frac{\delta\nu^2}{x})}$$

Under assumption (C.22), note that both roots are real. Furthermore, both roots are strictly positive and distinct for any $x > \delta\nu^2/\lambda$. In such case, we have $f(q,x) > 0$ for all $q \in (q_-(x), q_+(x))$. Thus, the inequality (C.26), combined with the fact that

$$\underline{q}^*_t \in \Big(q_-(\tilde{x}^*_\tau),\, q_+(\tilde{x}^*_\tau)\Big)$$

proves that $f(q^*_\tau; \tilde{x}^*_\tau) > 0$.

Finally, we may define a sequence of stopping times as follows. Let $\tau_0 := \tau$ and define recursively

$$\tau_{k+1} := \Big\{ t > \tau_k : q^*_t \le \underline{q}^*_t := \frac{\delta + \lambda\delta^{-1} - \nu^2}{2(\lambda - \delta\nu^2/\tilde{x}^*_t)} \Big\}.$$

The same method above can used to prove that $dq^*_{\tau_k} > 0$ for any k, which implies $\tau_{k+1} > \tau_k$ almost-surely. Then, in each time interval (τ_k, τ_{k+1}), we have that $q^*_t \ge \underline{q}^*_t$. Furthermore, we have $\underline{q}^*_t > \frac{\delta+\lambda\delta^{-1}-\nu^2}{2\lambda} > 0$, following the proof method above. By piecing together the sequences of stopped processes, this completes the proof that $(q^*_t)_{t\ge 0} > 0$ almost-surely, as long as $\psi^*_{\tau_k} = 0$ for each k. $\square$

C.2 Debt overhang as a "stabilizing force"

In this section, we sketch an economy where firms face an investment problem, subject to neoclassical adjustment costs and debt-overhang. The result is a version of Q-theory, but with underinvestment. Because the predictions of this theory are so well-established, at some points we make reduced-form assumptions to simplify the analysis and illustrate our main points on stability.

Firms. There are a continuum of firms in each location n, each employing a linear technology with productivity a and capital as the sole input. The evolution of firm-level capital is

$$dk^{(j)}_{n,t} = k^{(j)}_{n,t}[\iota^{(j)}_{n,t} - \kappa]dt + k^{(j)}_{n,t}\hat{\sigma}d\hat{B}^{(j)}_{n,t},$$

where ι is the endogenous investment rate, κ is the exogenous depreciation rate, and $B^{(j)}$ is an idiosyncratic Brownian shock. The cost of making investment ιk is given by $\Phi(\iota)k$, where $\Phi(\cdot)$ is a convex adjustment cost function. Thus, the investment-production block has the standard homogeneity property in capital.

For this section only, we denote by $q^{(j)}_{n,t}$ the location-n average value of capital to equity, i.e. "average Q" (this is not be the same as the price-dividend ratio that is called "q" in the main text, because the dividend is output minus investment). Thus, the value of firm j is given by $q^{(j)}_{n,t}k^{(j)}_{n,t}$.

We also assume that all firms have long-term debt outstanding, in fact a perpetuity with a fixed and continuously-paid coupon as in Leland (1994) and its descendent papers, without microfounding the reasons for why (e.g., debt tax shield), as this is unimportant. Furthermore, to keep things simple, we assume existing firms can never issue new debt. Finally, firms default optimally, subject to some default costs that are proportional to the firm's capital (these can be redistributed to households to create no deadweight loss). Under these conditions, a typical finding is (see for example Hennessy, 2004, Proposition 2)

$$\tilde{q}^{(j)}_{n,t} := \text{marginal value of capital to equity} < \text{average value of capital to equity} = q^{(j)}_{n,t}.$$

Moreover, essentially by definition of $\tilde{q}$, the optimal investment satisfies $\tilde{q}^{(j)}_{n,t} = \Phi'(\iota^{(j)}_{n,t})$ (see for example Hennessy, 2004, equation 11). Thus, we see that $q^{(j)}_{n,t} > \Phi'(\iota^{(j)}_{n,t})$. The lack of equality here measures the deviation from neoclassical Q-theory.

Despite this deviation, we have the following property. Since $q^{(j)}_{n,t}$ increases with $\tilde{q}^{(j)}_{n,t} = \Phi'(\iota^{(j)}_{n,t})$, and since Φ is a convex function, we have $\iota^{(j)}_{n,t}$ increasing in $q^{(j)}_{n,t}$. We furthermore make the reduced-form assumption that $\iota^{(j)}_{n,t} = \iota(q^{(j)}_{n,t})$ for some univariate increasing function $\iota(\cdot)$. This assumption is quite benign as it is typically satisfied in applications, because $\tilde{q}^{(j)}_{n,t}$, hence $q^{(j)}_{n,t}$, are typically monotonic functions of the underlying firm-level state (e.g., leverage ratio).

In summary, we have the following two firm-level properties under debt overhang:

$$q^{(j)}_{n,t} > \Phi'(\iota^{(j)}_{n,t}) \tag{C.27}$$

$$\iota'(q^{(j)}_{n,t}) > 0. \tag{C.28}$$

Condition (C.27) captures the specific debt-overhang mechanism, whereas condition (C.28) is much more general and applies in almost any investment model. With a more general contractual structure, DeMarzo et al. (2012) also obtains these two results.

Aggregation. We now make two assumptions that are mainly for tractability in aggregation. First, when a firm defaults and exits, it is replaced by another firm with the same identity j that inherits the defaulting capital stock. We assume this new entrant issues new debt is such that the aggregate location-n value of debt outstanding is always a constant fraction of total location-n capital; i.e., total location-n value of debt is always $\beta k_{n,t}$. Alternatively, this proportionality of aggregate debt to capital could be ensured by augmenting the model with a time-varying exogenous exit rate, but allowing new entrants to issue debt in an optimal way. Either way, this set of assumptions implies it suffices to study equity.

Second, we make assumptions to avoid studying the full cross-sectional distribution of firms within a location. We assume that properties (C.27)-(C.28) also hold in the aggregate at each location, and we presume a certain approximate aggregation on investment and investment costs. In particular, let us define the appropriate aggregates, for capital, average Q, and investment:

$$k_{n,t} := \int k^{(j)}_{n,t} dj$$

$$q_{n,t} := \frac{1}{k_{n,t}} \int q^{(j)}_{n,t} k^{(j)}_{n,t} dj$$

$$\iota_{n,t} := \frac{1}{k_{n,t}} \int \iota(q^{(j)}_{n,t}) k^{(j)}_{n,t} dj.$$

As an approximation, we assume the existence of functions $(\bar{\iota}, \bar{\Phi})$ such that the following hold:

$$\bar{\iota}(q_{n,t}) \approx \int k^{(j)}_{n,t} \iota(q^{(j)}_{n,t}) dj \tag{C.29}$$

$$k_{n,t} \bar{\Phi}(\bar{\iota}(q_{n,t})) \approx \int k^{(j)}_{n,t} \bar{\Phi}(\bar{\iota}(q^{(j)}_{n,t})) dj. \tag{C.30}$$

The nature of these approximations is to say that aggregate location-n investment is solely a function of aggregate average Q, rather than the full cross-sectional distribution of average Q's. Furthermore, we assume the following aggregate versions of properties (C.27)-(C.28), i.e.,

$$q_{n,t} > \bar{\Phi}'(\bar{\iota}_{n,t}) \tag{C.31}$$

$$\bar{\iota}'(q_{n,t}) > 0. \tag{C.32}$$

We conjecture these properties would go through in a full analysis of equilibrium using the cross-sectional distribution of firm size and Q, but this is beyond the scope of this paper. As we make these aggregation approximations, note that we also assume the functions $(\bar{\iota}, \bar{\Phi})$ are independent of location n.

Stability. Now, we can proceed to study stability. The aggregate portfolio of location-n firms' liabilities (debt plus equity) has value $(\beta + q_{n,t})k_{n,t}$, which is a claim to the profits $\int (a - \Phi(\iota_{n,t}^{(j)}))k_{n,t}^{(j)}dj$. Based on approximation (C.30), this aggregate profit can be approximately written $(a - \bar{\Phi}(\bar{\iota}(q_{n,t})))k_{n,t}$. Furthermore, the return on this portfolio is deterministic, given that all fundamental shocks are idiosyncratic (hence defaults are idiosyncratic), and thus the return must equal the riskless bond return r_t in equilibrium. Thus, $q_{n,t}$ evolves deterministically, and the (approximate) valuation equation states

$$\frac{a - \bar{\Phi}(\bar{\iota}(q_{n,t}))}{q_{n,t}} + \bar{\iota}(q_{n,t}) - \kappa + \frac{\dot{q}_{n,t}}{q_{n,t}} = r_t. \tag{C.33}$$

Lemma C.2. *Suppose the number of locations N is large enough, that approximations (C.29)-(C.30) hold, and that properties (C.31)-(C.32) hold with sufficient gaps between the left- and right-hand-sides (i.e., under-investment is large enough). Then, the equilibrium of the model with debt overhang is locally-stable.*

Proof of Lemma C.2. We start with approximate valuation equation (C.33). Differentiate $\dot{q}_{n,t}$ with respect to $q_{n,t}$ and $q_{-n,t}$ to obtain

$$\frac{d\dot{q}_{n,t}}{dq_{n,t}} = r_t + \kappa - \bar{\iota}(q_{n,t}) + \bar{\Phi}'(\bar{\iota}(q_{n,t}))\bar{\iota}'(q_{n,t}) - q_{n,t}\bar{\iota}'(q_{n,t}) + q_{n,t}\frac{dr_t}{dq_{n,t}}$$
$$\frac{d\dot{q}_{n,t}}{dq_{-n,t}} = q_{n,t}\frac{dr_t}{dq_{-n,t}}.$$

We study these equations in the limit $N \to \infty$, which suffices, because the lemma allows us to later make N large enough.

As $N \to \infty$, one can show that

$$r_t \to \delta - \kappa + \lim_{N\to\infty} \sum_{n=1}^{N} \frac{k_{n,t}}{\sum_{i=1}^{N} k_{i,t}} \bar{\iota}(q_{n,t}),$$

which has zero derivative with respect to $q_{i,t}$ for any i. Substituting this result for r_t, we obtain $d\dot{q}_{n,t}/dq_{-n,t} = 0$ and

$$\frac{d\dot{q}_{n,t}}{dq_{n,t}} = \delta + \underbrace{\lim_{N\to\infty} \sum_{m=1}^{N} \frac{k_{m,t}}{\sum_{i=1}^{N} k_{i,t}} \bar{\iota}(q_{m,t}) - \bar{\iota}(q_{n,t})}_{=0 \text{ in steady state}} - \left[q_{n,t} - \bar{\Phi}'(\bar{\iota}(q_{n,t}))\right]\bar{\iota}'(q_{n,t}).$$

The fact that the middle terms net out to zero in steady state is a consequence of the fact that $dk_{n,t} = k_{n,t}[\bar{\iota}(q_{n,t}) - \kappa]dt$, and all locations must experience the same growth rate $\bar{\iota}(q_{n,t}) - \kappa$ in steady state. Thus, we have $d\dot{q}_{n,t}/dq_{n,t} < 0$, hence local stability by $d\dot{q}_{n,t}/dq_{-n,t} = 0$, if and only if

$$\big[q_{n,t} - \bar{\Phi}'(\bar{\iota}(q_{n,t}))\big]\bar{\iota}'(q_{n,t}) > \delta.$$

This is true if properties (C.31)-(C.32) hold with sufficient gaps, as assumed. □

C.3 Creative destruction as a "stabilizing force"

In this section, we consider another model that allows multiplicity. We show how an overlapping generations (OLG) "perpetual youth" economy – built upon Blanchard (1985) – augmented with a particular type of creative destruction – similar to Gârleanu and Panageas (2020) – creates a stabilizing force upon which extrinsic shocks can be layered. In particular, if new firm creation is more intense when asset valuations are low, the economy possesses a natural stabilizing force. A possible rationale for this feature is that when capital asset valuations are low, they make labor look relatively attractive, which offers a robust outside option for those new entrepreneurs willing to enter. The contribution relative to Gârleanu and Panageas (2020) is to show how this is possible with an arbitrary number of assets (corresponding to the N locations) whose markets are, in addition, not integrated.

Cohorts, Endowments, Markets. In this model, all agents face a constant hazard rate of death $\beta > 0$, with all dying agents replaced by newborns (in the same location), so that population size is constant at 1. To keep matters simple, assume all locations have identical constant endowment growth rates and no shocks. That said, the endowment growth of an individual agent differs from the aggregate growth rate; this is the crucial ingredient in this model.

In particular, we assume some amount of *creative destruction*. The endowments of living agents decay at rate $\kappa_{n,t}$ (obsolescence rate), while newborn agents arrive to the economy with new trees of total size $\kappa_{n,t} + g$ (or, in per capita units, their individual trees are $(\kappa_{n,t} + g)/\beta$ in size). Specifically, the time-t endowment accruing to location-n agents born at time $s \leq t$ is

$$y_{n,t}^{(s)} = y_{n,t}(\kappa_{n,s} + g)\exp\Big[-\int_s^t(\kappa_{n,u} + g)du\Big].$$

Note that the aggregate endowment follows

$$dy_{n,t} = d\Big(\int_{-\infty}^t y_{n,t}^{(s)}ds\Big) = y_{n,t}^{(t)}dt + \int_{-\infty}^t dy_{n,t}^{(s)}ds = \underbrace{y_{n,t}(\kappa_{n,t} + g)dt}_{\text{newborn entry}} - \underbrace{y_{n,t}\kappa_{n,t}dt}_{\text{obsolescence}} = y_{n,t}g\,dt.$$

For now, we leave $\kappa_{n,t}$ unspecified, but note that its formulation will be the determinant of whether multiplicity is possible or not.

Agents can only trade in financial markets while alive. In addition to the tradability of claims to local endowments, agents can access a market for annuities that insures their death hazard and provides a stream of $\beta w_{n,t}^{(s)}$ of income per unit of time, where $w_{n,t}^{(s)}$ is the wealth of a location-n agent born at time $s \leq t$. This assumption is standard in perpetual youth models.

Solution. Under these assumptions, one can show that agents consume $\delta + \beta$ fraction of their wealth, so that the aggregate wealth condition (A.19) is replaced by

$$\sum_{n=1}^N \alpha_n q_{n,t} = (\delta + \beta)^{-1},$$

where $q_{n,t}$ is the (aggregated across cohorts) location-n valuation ratio. Let $\xi_{n,t}$ denote the location-n state-price density, which follows

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \pi_{n,t} dZ_{n,t}\Big].$$

We continue to examine a bubble-free equilibrium, so that

$$q_{n,t} = \mathbb{E}_t\Big[\int_t^\infty \frac{\xi_{n,\tau}}{\xi_{n,t}} \frac{y_{n,\tau}^{(s)}}{y_{n,t}^{(s)}} d\tau\Big] \quad \text{(for any birth-date } s \le t\text{, this yields the same answer).}$$

Critically, this valuation does not incorporate wealth gains due to entry of future newborns (i.e., this is the value of alive firms). The dynamic counterpart of this valuation equation is, for some diffusion coefficient $\sigma_{n,t}^q$,

$$\frac{dq_{n,t}}{q_{n,t}} = \Big[r_t + \kappa_{n,t} - \frac{1}{q_{n,t}} + \sigma_{n,t}^q \pi_{n,t}\Big]dt + \sigma_{n,t}^q dZ_{n,t}. \tag{C.34}$$

The equilibrium riskless rate is obtained as follows. The goods market is integrated across locations, so the market clearing condition is given by

$$Y_t = \sum_{n=1}^N y_{n,t} = \sum_{n=1}^N \int_{-\infty}^t \beta e^{-\beta(t-s)} c_{n,t}^{(s)} ds.$$

Optimal consumption dynamics for alive agents are

$$\frac{dc_{n,t}^{(s)}}{c_{n,t}^{(s)}} = \Big[r_t - \delta + \pi_{n,t}^2\Big]dt + \pi_{n,t} dZ_{n,t},$$

whereas newborn agents consume

$$\beta c_{n,t}^{(t)} = \underbrace{(\delta+\beta)}_{\substack{\text{cons-wealth}\\ \text{ratio}}} \times \underbrace{(\kappa_{n,t} + g) y_{n,t} q_{n,t}}_{\text{newborn wealth}}.$$

Applying Itô's formula to goods market clearing, and using these results, we obtain

$$r_t = \delta + \beta - \sum_{n=1}^N x_{n,t} \pi_{n,t}^2 - (\delta+\beta) \sum_{n=1}^N \alpha_n q_{n,t} \kappa_{n,t}. \tag{C.35}$$

Stability. To see how the stabilizing force works, it is instructive to once again study the deterministic equilibrium in which extrinsic shocks have no volatility. Substituting (C.35) into (C.34) with $\sigma_{n,t}^q = 0$, we obtain

$$\dot{q}_{n,t} = \underbrace{-1 + (\delta+\beta) q_{n,t}}_{\text{unstable component}} - \underbrace{\Big[(\delta+\beta)\sum_{i=1}^N \alpha_i q_{i,t} \kappa_{i,t} - \kappa_{n,t}\Big] q_{n,t}}_{\text{stabilizing force}} \quad \text{when} \quad \sigma_{i,t}^q = 0 \quad \forall i. \tag{C.36}$$

The first piece is the unstable component, propelling valuations further and further away from the "steady state" value $(\delta+\beta)^{-1}$. The second piece—capturing the relative amount of creative destruction in location n—is the stabilizing force.

Based on equation (C.36), we claim that if $\kappa_{n,t}$ decreases sufficiently rapidly as $q_{n,t}$ increases, then valuation dynamics are stable. Let $\kappa_{n,t} = \kappa(q_{n,t})$ for a decreasing function $\kappa(\cdot)$. Denote the steady-state mean and sensitivity of this function by $\bar{\kappa} := \kappa((\delta+\beta)^{-1})$ and $\lambda := -\kappa'((\delta+\beta)^{-1})$, respectively. Then, compute

$$\frac{\partial \dot{q}_n}{\partial q_m}\Big|_{q_i=(\delta+\beta)^{-1}\,\forall\, i} = \begin{cases} \delta+\beta-\lambda(\delta+\beta)^{-1}(1-\alpha_n)-\alpha_n\bar{\kappa}, & \text{if } m=n; \\ \lambda(\delta+\beta)^{-1}\alpha_m-\alpha_m\bar{\kappa}, & \text{if } m\neq n. \end{cases}$$

Construct the steady-state Jacobian matrix as

$$J := \Big[\frac{\partial \dot{q}_n}{\partial q_m}\Big|_{q_i=(\delta+\beta)^{-1}\,\forall\, i}\Big]_{1\le n,m\le N}. \tag{C.37}$$

Local stability of the steady-state can be determined by the eigenvalues of J. By the Gershgorin circle theorem, all of these eigenvalues have strictly negative real parts if J has negative diagonal elements and is diagonally dominant. This is easily guaranteed by making $\bar{\kappa}$ and λ large enough, meaning the amount of creative destruction and its sensitivity to prices are both large enough. The result is summarized in the following lemma, with the proof omitted.

Lemma C.3. *Assume $\bar{\kappa} > \delta+\beta$ and $\lambda > (\delta+\beta)\bar{\kappa}$. Then, all eigenvalues of J have strictly negative real parts. Consequently, the equilibrium of the creative destruction model is locally stable.*

D Example: sunspot fluctuations in the aggregate valuation

Most of the paper focuses on redistributive fluctuations. Here, we also present an example in which the aggregate valuation can be subject to self-fulfilling fluctuations. The results of Theorem 1 and Lemmas 1-2 imply that an indeterminate aggregate valuation requires $\rho < 1$ and a sufficiently large growth-valuation link parameter λ.

We present this example in a one-location economy ($N = 1$), so without loss of generality we may also shut down the idiosyncratic fundamental shock ($\hat{\nu} = 0$). Eq. (A.24) contains the aggregate valuation dynamics dQ_t in general. Substituting $\pi_t = 0$ due to Eq. (A.17), as well as the expression for r_t in (A.14) and the expression for growth g_t in (7), the aggregate valuation ratio Q_t satisfies

$$\begin{aligned} dQ_t = Q_t\Big[\delta+(\rho-1)(g-\lambda q^*)+(\rho-1)\lambda Q_t-\frac{1}{2}\rho(\rho-1)\nu^2-\frac{1}{Q_t}+(\rho-1)\nu\varsigma_t^Q\Big]dt \\ + Q_t\Big[\varsigma_t^Q dB_t+\sigma_t^Q dZ_t\Big]. \end{aligned}$$

Following the constructions in Propositions 1-2, let us conjecture an equilibrium with $\varsigma_t^Q = 0$. In that case, and recalling that $q^* = \delta+(\rho-1)g-\frac{1}{2}\rho(\rho-1)\nu^2$, we may rewrite the dynamics as

$$\begin{aligned} dQ_t &= D(Q_t)dt + Q_t\sigma_t^Q dZ_t \\ \text{where}\quad D(Q) &:= -1+Q\Big[\frac{1}{q^*}+(1-\rho)\lambda q^*\Big]-(1-\rho)\lambda Q^2 \\ &= -(Q-q^*)\Big(\lambda(1-\rho)Q-\frac{1}{q^*}\Big). \end{aligned} \tag{D.1}$$

The only question for whether or not we have equilibrium is whether or not the dynamics in (D.1) keep Q_t positive and bounded. Basically, this boils down to the properties of the function $D(\cdot)$, as well as how σ_t^Q is specified.

We require $\rho < 1$ and $\lambda > \frac{1}{(1-\rho)(q^*)^2}$. In that case, the shape of the function $D(\cdot)$ is an inverse-U with two steady states, q^* and $q^{**} := \frac{1}{\lambda(1-\rho)q^*} < q^*$. The larger steady state is locally stable, since $D'(q^*) < 0$—exactly as in Theorem 1. The smaller steady state q^{**} is unstable. Therefore, the function $D(\cdot)$ is positive for $Q \in (q^{**}, q^*)$ and negative for $Q > q^*$. The idea is then to specify σ_t^Q to keep Q_t in the region (q^{**}, ∞). The following formal result explains how this can be done. We omit the proof as it is based on standard boundary classification results for one-dimensional SDEs.

Proposition D.1. *Suppose $N = 1$. Pick an interval $[\underline{q}, \bar{q}]$, where $q^{**} \le \underline{q} < q^* < \bar{q}$. Pick a bounded function V such that $V(q) > 0$ on $(\underline{q}, \bar{q})$ and such that*

$$\lim_{q \to \bar{q}} \frac{V(q)^2}{\bar{q} - q} < -2D(\bar{q}) \quad \textit{and} \quad \lim_{q \to \underline{q}} \frac{V(q)^2}{q - \underline{q}} < 2D(\underline{q}). \tag{D.2}$$

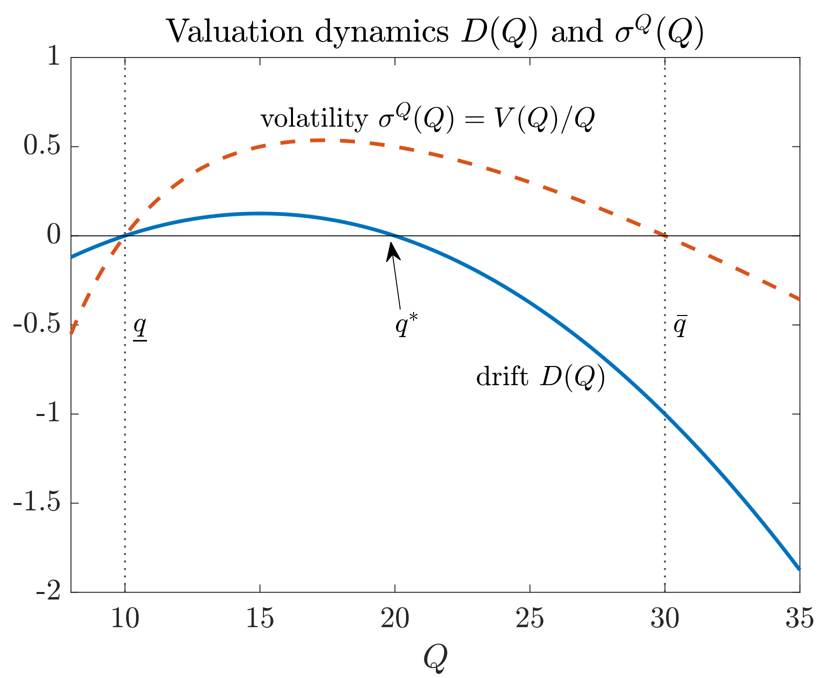
Then, there exists an equilibrium where Q_t follows

$$dQ_t = D(Q_t)dt + V(Q_t)dZ_t$$

and remains forever inside $[\underline{q}, \bar{q}]$.

Figure D.1 presents an example of such a construction, where we have used the volatility function $V(q) = 0.1(q - \underline{q})(\bar{q} - q)$, which satisfies condition (D.2). You can see that volatility vanishes at the points $\underline{q}$ and $\bar{q}$, which allows the drift $D(Q)$ to take over at those points, inducing the valuation to mean-revert to steady state.

Figure D.1: Aggregate valuation dynamics.



Notes. Parameters are $\delta = 0.05$, $g = 0$, $\nu = 0$, $\rho = 0.5$, $\lambda = \frac{2}{(1-\rho)(q^*)^2}$, and $\bar{q} = 1.5q^*$.

E International model of Section 5.2

Derivation of equilibrium. As before, let $Y_t := \sum_{n=1}^N y_{n,t}$ be aggregate tradable consumption, and define (tradable) consumption shares $x_{n,t} := c_{n,t}/Y_t$ and (tradable) endowment shares $\alpha_{n,t} := y_{n,t}/Y_t$. The country-$n$ state price density $\xi_{n,t}$ still evolves according to Eq. (A.1), repeated here for convenience

$$\frac{d\xi_{n,t}}{\xi_{n,t}} = -r_t dt - \eta_t dB_t - \hat{\eta}_t \cdot d\hat{B}_t - \pi_{n,t} \cdot dZ_t.$$

The representative agent of country n maximizes lifetime utility (26) subject to the lifetime budget constraint, i.e.,

$$\begin{aligned} &\max_{c_n,\hat{c}_n,w_n} \mathbb{E}_0\Big[\int_0^\infty e^{-\delta t}\Big(\phi\log(c_{n,t}) + (1-\phi)\log(\hat{c}_{n,t})\Big)dt\Big] \\ &\text{s.t.} \quad w_{n,0} = \mathbb{E}_0\Big[\int_0^\infty \frac{\xi_{n,t}}{\xi_{n,0}}(c_{n,t} + p_{n,t}\hat{c}_{n,t})dt\Big]. \end{aligned} \tag{E.1}$$

Solving this maximization problem delivers FOCs $e^{-\delta t}\phi c_{n,t}^{-1} = \xi_{n,t}$ and $e^{-\delta t}(1-\phi)\hat{c}_{n,t}^{-1} = \xi_{n,t}p_{n,t}$, which together imply the expenditure shares

$$c_{n,t} = \phi P_{n,t}C_{n,t} \quad \text{and} \quad p_{n,t}\hat{c}_{n,t} = (1-\phi)P_{n,t}C_{n,t}. \tag{E.2}$$

Next, substitute these FOCs back into the budget constraint (E.1) to get $c_{n,0} + p_{n,0}\hat{c}_{n,0} = \delta w_{n,0}$. Using the definition of the price and quantity index $P_{n,t}C_{n,t} = c_{n,t} + p_{n,t}\hat{c}_{n,t}$, we obtain the expenditure rule

$$P_{n,t}C_{n,t} = \delta w_{n,t}. \tag{E.3}$$

Therefore, the optimal dynamics of non-tradable consumption $c_{n,t}$, expenditure $P_{n,t}C_{n,t}$, and wealth $w_{n,t}$ all take the same geometric form, namely

$$\frac{dc_{n,t}}{c_{n,t}} = \frac{dw_{n,t}}{w_{n,t}} = \Big[r_t - \delta + \eta_t^2 + \|\hat{\eta}_t\|^2 + \|\pi_{n,t}\|^2\Big]dt + \eta_t dB_t + \hat{\eta}_t \cdot d\hat{B}_t + \pi_{n,t} \cdot dZ_t. \tag{E.4}$$

As in the baseline model, using $\sum_{n=1}^N dc_{n,t} = dY_t$ and matching drifts and diffusions, we obtain the interest rate in Eq. (A.14) and risk prices in Eqs. (A.15)-(A.17), all repeated here for convenience

$$r_t = \delta + g_t - \nu^2 - \sum_{n=1}^N x_{n,t}\|\pi_{n,t}\|^2, \quad \text{and} \quad \eta_t = \nu, \quad \text{and} \quad \hat{\eta}_t = 0, \quad \text{and} \quad \sum_{n=1}^N x_{n,t}\pi_{n,t} = 0. \tag{E.5}$$

Therefore, the dynamics of wealth shares $x_{n,t}$ are identical to the baseline model Eq. (A.18). Similarly, endowment shares $\alpha_{n,t}$ still evolve according to Eq. (A.4). Finally, use the tradable expenditure share rule to write aggregate wealth as

$$\sum_{n=1}^N w_{n,t} = \sum_{n=1}^N \frac{c_{n,t}}{\phi\delta} = \frac{Y_t}{\phi\delta}. \tag{E.6}$$

So far, this is nearly identical to the baseline model. The step that diverges from the baseline model, which we tackle next, regards the local equity pricing equation.

The return on local equity $dR_{n,t}$ is defined by

$$dR_{n,t} := \frac{1}{q_{n,t}}dt + \frac{d(q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t}))}{q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})},$$

where the valuation ratio $q_{n,t}$ has dynamics of the form

$$\frac{dq_{n,t}}{q_{n,t}} = \mu^q_{n,t}dt + \varsigma^q_{n,t}dB_t + \hat{\varsigma}^q_{n,t} \cdot d\hat{B}_t + \sigma^q_{n,t} \cdot dZ_t.$$

The dividend on equity can be written, using the result from Eqs. (E.2)-(E.3) that $p_{n,t}\hat{y}_{n,t} = \frac{1-\phi}{\phi}c_{n,t}$,

$$y_{n,t} + p_{n,t}\hat{y}_{n,t} = \frac{Y_t}{\phi}\Big(\phi\alpha_{n,t} + (1-\phi)x_{n,t}\Big). \tag{E.7}$$

Apply Itô's formula to $d(q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t}))$, using (E.5) and (E.7), to obtain

$$\begin{aligned} dR_{n,t} = {} & \frac{1}{q_{n,t}}dt + \mu^q_{n,t}dt + \varsigma^q_{n,t}dB_t + \hat{\varsigma}^q_{n,t} \cdot d\hat{B}_t + \sigma^q_{n,t} \cdot dZ_t \\ & + \frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}\Big([g_{n,t} + \nu\varsigma^q_{n,t} + \hat{\nu}_{n,t} \cdot \hat{\varsigma}^q_{n,t}]dt + \nu dB_t + \hat{\nu}_{n,t} \cdot d\hat{B}_t\Big) \\ & + \frac{(1-\phi)x_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}\Big([r_t - \delta + \nu^2 + \|\pi_{n,t}\|^2 + \nu\varsigma^q_{n,t} + \pi_{n,t} \cdot \sigma^q_{n,t}]dt + \nu dB_t + \pi_{n,t} \cdot dZ_t\Big) \end{aligned} \tag{E.8}$$

Consequently, the no-arbitrage pricing equation is (after substituting the equilibrium risk prices and doing extensive algebra)

$$\mu^q_{n,t} = \delta - \frac{1}{q_{n,t}} + \Phi_{n,t}\Big(r_t - \delta + \nu^2 + \pi_{n,t} \cdot \sigma^q_{n,t} - g_{n,t} - \hat{\nu}_{n,t} \cdot \hat{\varsigma}^q_{n,t}\Big), \tag{E.9}$$

where

$$\Phi_{n,t} := \frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}. \tag{E.10}$$

Note that Eq. (E.10) defines the endogenous value share of tradable output, relative to country-n total output. While $\Phi_{n,t} = \phi$ in a symmetric steady state, $\Phi_{n,t}$ moves around because non-tradable prices $p_{n,t}$ move around, in contrast to the constant tradable consumption share ϕ.

To connect the risk prices to the valuation dynamics, recall the dynamic budget constraint

$$dw_{n,t} = (w_{n,t}r_t - P_{n,t}C_{n,t})dt + \vartheta_{n,t}(\eta_t dt + dB_t) + \hat{\vartheta}_{n,t} \cdot (\hat{\eta}_t dt + d\hat{B}_t) + \theta_{n,t}(dR_{n,t} - r_t dt). \tag{E.11}$$

First, using local equity market clearing $\theta_{n,t} = q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})$ and matching the dZ_t loadings in Eq. (E.11) to those in Eq. (E.4), we have

$$\pi_{n,t} = \frac{q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})}{w_{n,t}}\Big(\sigma^q_{n,t} + \frac{p_{n,t}\hat{y}_{n,t}}{y_{n,t} + p_{n,t}\hat{y}_{n,t}}\pi_{n,t}\Big)$$

Solving the above equation for $\pi_{n,t}$, and using Eqs. (E.2)-(E.3) to simplify, we obtain

$$\pi_{n,t} = \delta\frac{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}{x_{n,t}(1-(1-\phi)\delta q_{n,t})}q_{n,t}\sigma^q_{n,t}. \tag{E.12}$$

Second, summing both Eqs. (E.11) and (E.4) over n, using $\eta_t = \nu$ and $\hat{\eta}_t = 0$, using futures market clearing conditions $\sum_{n=1}^N \vartheta_{n,t} = 0$ and $\sum_{n=1}^N \hat{\vartheta}_{n,t} = 0$, and using the aggregate wealth constraint (E.6), we obtain

$$\nu = \delta\sum_{n=1}^N \Big(\phi\alpha_{n,t} + (1-\phi)x_{n,t}\Big)q_{n,t}(\nu + \varsigma^q_{n,t})$$

$$0 = \sum_{n=1}^N \Big(\phi\alpha_{n,t} + (1-\phi)x_{n,t}\Big)q_{n,t}(\hat{\nu}_{n,t} + \hat{\varsigma}^q_{n,t})$$

Finally, an additional consequence of $\sum_{n=1}^{N} \vartheta_{n,t} = 0$ and $\sum_{n=1}^{N} \hat{\vartheta}_{n,t} = 0$ is the wealth aggregation $\sum_{n=1}^{N} w_{n,t} = \sum_{n=1}^{N} q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})$ condition. Combining this with (E.7), we can rewrite (E.6) as

$$\sum_{n=1}^{N} \Big(\phi \alpha_{n,t} + (1-\phi) x_{n,t}\Big) q_{n,t} = \delta^{-1}. \tag{E.13}$$

From Eq. (E.13), it is clear that the symmetric steady state must be $q_{n,t} = \delta^{-1}$. This completes the set of equilibrium equations, analogously to Appendix A.

Construction of redistributive sunspot equilibria. By $\sum_{n=1}^{N} dc_{n,t} = dY_t$ and (E.4), we have $0 = \sum_{n=1}^{N} x_{n,t}\pi_{n,t}$, as in the baseline model. To satisfy this constraint, we construct a candidate equilibrium as follows. Let M be an $N \times N$ matrix with $\mathrm{rank}(M) = N-1$ and unit-length columns, let $v^* := (v_1^*, \dots, v_N^*)' \geq 0$ be in the null-space of M, and let ψ_t be a positive scalar process (exactly as in Lemma 3). Set

$$\pi_{n,t} = \frac{\delta \psi_t}{x_{n,t}} v_n^* M e_n. \tag{E.14}$$

By Eq. (E.12), we then have

$$\frac{\phi \alpha_{n,t} + (1-\phi) x_{n,t}}{1 - (1-\phi)\delta q_{n,t}} q_{n,t} \sigma_{n,t}^{q} = \psi_t v_n^* M e_n, \tag{E.15}$$

which pins down valuation volatilities. The validity of this candidate equilibrium boils down to restrictions on the process ψ_t.

Growth-valuation link. As before, we assume

$$g_{n,t} = g + \lambda\big(q_{n,t} - \delta^{-1}\big). \tag{E.16}$$

Unlike the baseline model, this growth-valuation link makes aggregate growth time-varying with non-tradables. Indeed, $g_t = \sum_{i=1}^{N} \alpha_{i,t} g_{i,t} = g + \lambda\big(Q_t - \delta^{-1}\big)$, where

$$Q_t := \sum_{i=1}^{N} \alpha_{i,t} q_{i,t}$$

is the endowment-weighted aggregate valuation. To the extent that $x_{n,t}$ differs from $\alpha_{n,t}$, then Q_t will differ from $\sum_{i=1}^{N} (\phi \alpha_{i,t} + (1-\phi) x_{i,t}) q_{i,t} = \delta^{-1}$. This makes the stability analysis below significantly more complex.

Stability analysis for $\hat{v} = 0$ case. In this model, we require $\hat{v} = 0$ (no idiosyncratic fundamental shocks) in order to have self-fulfilling equilibria. The reason, which will become clear below, is that the stability properties of this model depend on the endowment shares $(\alpha_n)_{n=1}^{N}$ remaining relatively close to symmetric. As before, the key is to analyze the model dynamics—namely the dynamical system for $(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{q}) := (x_n, \alpha_n, q_n)_{n=1}^{N}$—when $\psi_t = 0$. The question is whether, when this sunspot volatility vanishes, the economy remains stable and does not diverge from steady state.

First, when $\psi_t = 0$, the sunspot risk prices $\pi_{n,t} = 0$ for all n, so the wealth shares $x_{n,t}$ remain constant. So we can eliminate $\boldsymbol{x} = (x_n)_{n=1}^{N}$ from consideration in our dynamical system. Second, the endowment shares $\boldsymbol{\alpha}$ follow, after substituting $\hat{v} = 0$ and Eq. (E.16) into (A.4),

$$\dot{\alpha}_n = \lambda \alpha_n (q_n - Q), \tag{E.17}$$

of in vector notation

$$\dot{\alpha}_n = F_n(\boldsymbol{\alpha}, \boldsymbol{q}) := \lambda e_n \cdot \boldsymbol{\alpha}(e_n \cdot \boldsymbol{q} - \boldsymbol{\alpha} \cdot \boldsymbol{q}),$$

where e_n is the nth elementary vector. Third, substitute $\hat{\nu} = \hat{\varsigma}^q_{n,t} = 0$, $r_t = \delta + g_t - \nu^2 - \sum_{n=1}^N x_{n,t}\|\pi_{n,t}\|^2$, and Eqs. (E.14)-(E.15)-(E.16) into Eq. (E.9) to get

$$\begin{aligned} q_{n,t}\mu^q_{n,t} &= -1 + \Big(\delta + \Phi_{n,t}\lambda Q_t\Big) q_{n,t} - \Phi_{n,t}\lambda q^2_{n,t} \\ &\quad + \Phi^2_{n,t}\frac{\delta(\psi_t v^*_n)^2}{\phi\alpha_{n,t}x_{n,t}}[1-(1-\phi)\delta q_{n,t}] - \Phi_{n,t}\Big(\delta^2\psi_t^2\sum_{i=1}^N \frac{(v^*_i)^2}{x_{i,t}}\Big) q_{n,t}. \end{aligned} \tag{E.18}$$

When $\psi_t = 0$, the entire second line of Eq. (E.18) vanishes, and

$$\dot{q}_n = -1 + \Big(\delta + \frac{\phi\alpha_n}{\phi\alpha_n + (1-\phi)x_n}\lambda Q\Big) q_n - \frac{\phi\alpha_n}{\phi\alpha_n + (1-\phi)x_n}\lambda q_n^2, \tag{E.19}$$

or in vector notation

$$\dot{q}_n = D_n(\boldsymbol{\alpha}, \boldsymbol{q}) := -1 + \Big(\delta + \frac{\phi e_n \cdot \boldsymbol{\alpha}}{\phi e_n \cdot \boldsymbol{\alpha} + (1-\phi)x_n}\lambda \boldsymbol{\alpha}\cdot\boldsymbol{q}\Big) e_n \cdot \boldsymbol{q} - \frac{\phi e_n \cdot \boldsymbol{\alpha}}{\phi e_n \cdot \boldsymbol{\alpha} + (1-\phi)x_n}\lambda (e_n \cdot \boldsymbol{q})^2.$$

The steady state of the deterministic system defined by (E.17) and (E.19) is $\boldsymbol{q} = \delta^{-1}\mathbf{1}_N$. At this steady state, $\boldsymbol{\alpha}$ can take any value in the unit simplex.

The $2N \times 2N$ Jacobian matrix for $(\dot{\alpha}_n, \dot{q}_n)_{n=1}^N$ at this steady state is defined as

$$J := \begin{bmatrix} \big[\nabla_{\boldsymbol{\alpha}} F_n\big]_{1\le n\le N} & \big[\nabla_{\boldsymbol{q}} F_n\big]_{1\le n\le N} \\ \big[\nabla_{\boldsymbol{\alpha}} D_n\big]_{1\le n\le N} & \big[\nabla_{\boldsymbol{q}} D_n\big]_{1\le n\le N} \end{bmatrix}\Bigg|_{ss}$$

Computing these derivatives, we have

$$\begin{aligned} J &= \begin{bmatrix} \big[-\lambda\delta^{-1}\alpha_n\mathbf{1}'_N\big]_{1\le n\le N} & \big[\lambda\alpha_n(e_n-\boldsymbol{\alpha})'\big]_{1\le n\le N} \\ \big[\Phi_n\lambda\delta^{-2}\mathbf{1}'_N\big]_{1\le n\le N} & \big[\delta e'_n - \Phi_n\lambda\delta^{-1}(e_n-\boldsymbol{\alpha})'\big]_{1\le n\le N} \end{bmatrix} \\ &= \begin{bmatrix} -\lambda\delta^{-1}\alpha_1 & \cdots & -\lambda\delta^{-1}\alpha_1 & \lambda\alpha_1(1-\alpha_1) & \cdots & -\lambda\alpha_1\alpha_N \\ \vdots & & \vdots & & \ddots & \\ -\lambda\delta^{-1}\alpha_N & \cdots & -\lambda\delta^{-1}\alpha_N & -\lambda\alpha_N\alpha_1 & \cdots & \lambda\alpha_N(1-\alpha_N) \\ \Phi_1\lambda\delta^{-2} & \cdots & \Phi_1\lambda\delta^{-2} & \delta - \Phi_1\lambda\delta^{-1}(1-\alpha_1) & \cdots & \alpha_N\Phi_1\lambda\delta^{-1} \\ \vdots & & \vdots & & \ddots & \\ \Phi_N\lambda\delta^{-2} & \cdots & \Phi_N\lambda\delta^{-2} & \alpha_1\Phi_N\lambda\delta^{-1} & \cdots & \delta - \Phi_N\lambda\delta^{-1}(1-\alpha_N) \end{bmatrix} \end{aligned}$$

Notice that the first N columns of J are repeated, so that J has N eigenvectors corresponding to a zero eigenvalue. This eigenspace (i.e., the kernel of J) spans exactly the possible values taken by $\boldsymbol{\alpha}$. By inspection, we see that J also has one eigenvalue equal to δ, since the sum of columns $N+1$ through $2N$ of the matrix $J - \delta I_N$ gives the zero vector. This positive eigenvalue corresponds to the fact that the aggregate wealth constraint (E.13) pins down a single linear combination of the valuations in $\boldsymbol{q}$, and so one dimension of the dynamics is necessarily unstable. Finally, by properties of the trace, the remaining $N-1$ eigenvalues of J must sum to

$$\epsilon_N := (N-1)\delta - \lambda\delta^{-1}\sum_{n=1}^N \Phi_n(1-\alpha_n).$$

A sufficient condition for J to have the desired local stability properties that admit multiplicity is for $\epsilon_N < 0$. Indeed, in such case at least one of the remaining eigenvalues must be negative, meaning that there exists at least one direction in which $\boldsymbol{q}$ can deviate from steady state and remain stable, returning to steady state asymptotically.

A necessary stability condition is thus

$$\lambda \sum_{n=1}^{N} \Phi_n(1-\alpha_n) > (N-1)\delta^2. \tag{E.20}$$

In particular, we know that parameters must be set so that λ is sufficiently larger than δ^2/ϕ. However, as the discussion above reveals, (E.20) only permits one direction of multiplicity and does not suffice to allow arbitrary deviations of $\boldsymbol{q}$ from steady state. In a 2-country model ($N = 2$), condition (E.20) is necessary and sufficient for stability. But in a general N-country model, if we want arbitrary deviations of $\boldsymbol{q}$ from steady state, we would need to verify that J indeed has $N-1$ stable eigenvalues.

To check this, we enforce the stronger condition if $N > 2$:

$$\lambda \Phi_n(1-\alpha_n) > \frac{N-1}{N}\delta^2, \quad \forall n. \tag{E.21}$$

Condition (E.21) suffices to allow arbitrary deviations of $\boldsymbol{q}$ from steady state within an $(N-1)$-dimensional manifold. Since (E.21) depends on endogenous variables, specifically the distribution of endowment and wealth shares $(\boldsymbol{\alpha}, \boldsymbol{x})$, we must force volatility ψ_t to vanish in our stochastic simulations if ever $\Phi_n(1-\alpha_n)$ becomes too small.

These conditions suggest two other properties of our self-fulfilling equilibria in the international macro model. First, the presence of non-tradables (i.e., $\phi < 1$) limits to scope for self-fulfilling equilibria. Indeed, taking $\phi \to 1$ in condition (E.20) results in the condition $\lambda > \delta^2$, exactly as in the baseline model. All else equal, reducing the magnitude of ϕ reduces the left-hand-side of (E.20), thus tightening the stability condition.

Second, self-fulfilling equilibria require the endowment distribution to remain relatively close to symmetric. For example, in the 2-country case, condition (E.20) is necessary and sufficient for stability. The condition can be rewritten

$$(N=2): \quad \lambda \phi \alpha (1-\alpha) > \delta^2 \Big(\phi\alpha + (1-\phi)x\Big)\Big(\phi(1-\alpha) + (1-\phi)(1-x)\Big) \tag{E.22}$$

where α and x are the country-1 endowment and wealth shares. Notice that if α deviates too far from $1/2$, then (E.22) cannot hold. The levels of the endowment shares played no role in the baseline model with $\phi = 1$.

Finally, because the stability analysis above was purely local to steady state, we must force ψ_t to vanish if any q_n ever deviates too far from δ^{-1}. This is the counterpart to condition (P2) in Proposition 2 that makes volatilities vanish if the minimal or maximal valuation ever deviate too far. While we do not conduct a formal analysis of this "radius of convergence" (i.e., how far q_n can deviate and remain in the stable region), our simulations are set up conservatively to keep q_n within a tight range of δ^{-1}.

Figure E.1 displays the stability properties of this model via phase diagrams and some example paths from a 2-country model. Notice that as q_1 deviates too far from steady state, or if α_1 deviates too far from $1/2$, the paths become unstable and violate equilibrium requirements.

Figure E.2 displays a 100-year stochastic path of the 2-country model with sunspot shocks. Because $N = 2$, sunspot shock exposures are set so that only the first sunspot shock matters (i.e.,

$Me_1 \cdot dZ_t = Me_2 \cdot dZ_t = dZ_{1,t}$); this is the simplest design of a redistributive stochastic equilibrium. The volatility process ψ_t is chosen to follow a latent Feller square-root process that vanishes if any valuation $q_{n,t}$ deviates too far from steady state (more than 10%), or if condition (E.20) is violated. These conditions suffice to keep the prevent the economy from diverging. (However, note that if the simulation were to continue forever, the long-run amount of volatility would necessarily be zero, because the processes for $\alpha_{n,t}$ are non-stationary and converge to $\{0,1\}$ with probability 1.)

Finally, Figure E.3 displays some paths of relevant international macro-finance variables during this simulation.

The bottom two panels, concerning the yield-to-maturity gap between the two countries and the short-term uncovered interest parity deviation (UIP), are constructed as follows. Compute the price of a pure discount bond that pays off one unit of the country-n consumption basket:

$$b_{n,t\to T} := \mathbb{E}_t\Big[\frac{\xi_{n,T}}{\xi_{n,t}}\frac{P_{n,T}}{P_{n,t}}\Big].$$

Note that, due to the normalization by $P_{n,t}$, the price $b_{n,t\to T}$ is denominated in units of the country-n consumption basket. One can then show that the yield-to-maturity of this bond, $\text{YTM}_{n,t\to T} := -\frac{1}{T-t}\log(b_{n,t\to T})$, is given by[12]

$$\text{YTM}_{n,t\to t+\Delta} = \delta - \frac{1}{\Delta}\log\mathbb{E}_t\Big[\Big(\frac{x_{n,t}}{x_{n,t+\Delta}}\Big)^{\phi}\Big(\frac{\hat{y}_{n,t}}{\hat{y}_{n,t+\Delta}}\Big)^{1-\phi}\Big(\frac{Y_t}{Y_{t+\Delta}}\Big)^{\phi}\Big], \tag{E.23}$$

To produce a tractable version of this expression, we take the short-horizon limit $\Delta \to 0$. The result, after applying Itô's formula to the expectation, is

$$\lim_{\Delta\to 0}\text{YTM}_{n,t\to t+\Delta} = \Omega_t + (1-\phi)g_{n,t} + \frac{1}{2}\phi(1-\phi)(\delta\psi_t v_n^*)^2(\frac{1}{x_{n,t}})^2$$

where Ω_t represents terms that are independent of country n.

On the other hand, the UIP deviation is the expected excess carry return going long the country-j bond and short the country-i bond:

$$R^{i,j}_{t\to t+\Delta} := \text{YTM}_{j,t\to t+\Delta} - \text{YTM}_{i,t\to t+\Delta} + \frac{1}{\Delta}\mathbb{E}_t[\log\mathcal{E}^{i,j}_{t+\Delta} - \log\mathcal{E}^{i,j}_t]. \tag{E.24}$$

Once again, a simple expression emerges if we take the short-horizon limit $\Delta \to 0$. The result, after applying Itô's formula to the log exchange rate, and substituting the results above for yield-to-maturity, is

$$\lim_{\Delta\to 0} R^{i,j}_{t\to t+\Delta} = \frac{1}{2}(1+\phi)(1-\phi)(\delta\psi_t)^2\Big[v_j^*(\frac{1}{x_{j,t}})^2 - v_i^*(\frac{1}{x_{i,t}})^2\Big]$$

Thus, the model produces the unambiguous qualitative prediction that relatively poor countries (low x) have high UIP deviations, especially when volatility is high (high ψ). That said, as Figure E.3 shows, these UIP deviations are quantitatively very small, presumably due to the log preference assumption.

[12]To derive this equation, substitute the price index $P_{n,t} = \phi^{-1}(c_{n,t}/\hat{y}_{n,t})^{1-\phi}$ and use the optimal consumption rule $\xi_{n,t} = e^{-\delta t}\phi c_{n,t}^{-1}$.

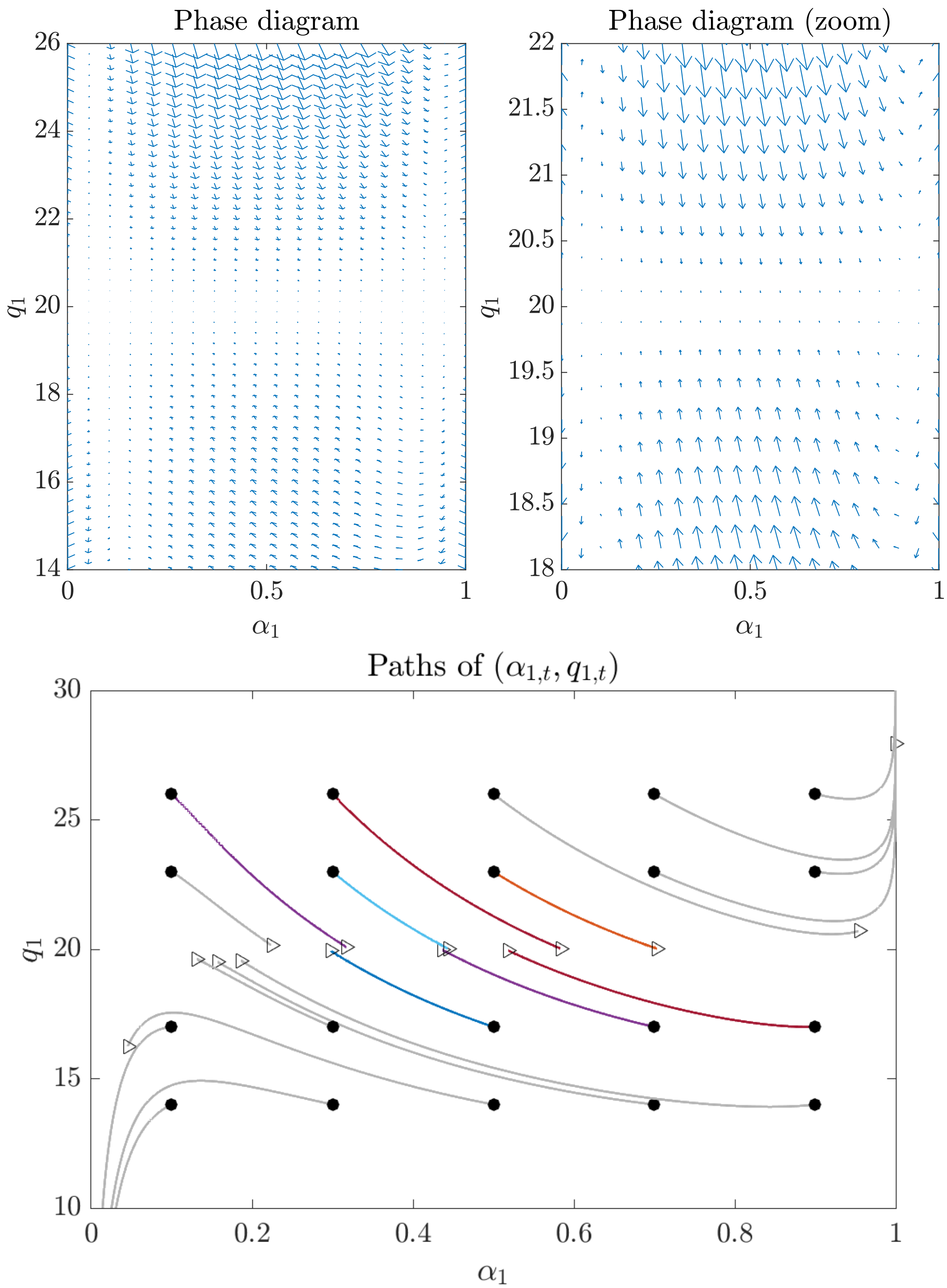


Figure E.1: Deterministic dynamics in a 2-country model.

Notes. The top left panel plots the phase diagram for (α_1, q_1), which form an autonomous ODE system in the deterministic $N = 2$ case. The top right panel zooms the left panel. The bottom panel displays some example paths starting from various initial levels of α_1 and q_1 (starting values indicated by black dots). In the bottom panel, the colored lines represent stable paths (which are valid equilibria), while the gray lines represent unstable, divergent paths (non-equilibria). Parameters: $\delta = 0.05$, $g = 0.02$, $\lambda = 0.0067$, $\phi = 0.75$. In all panels, the wealth distribution is symmetric: $x_1 = x_2 = 0.5$.

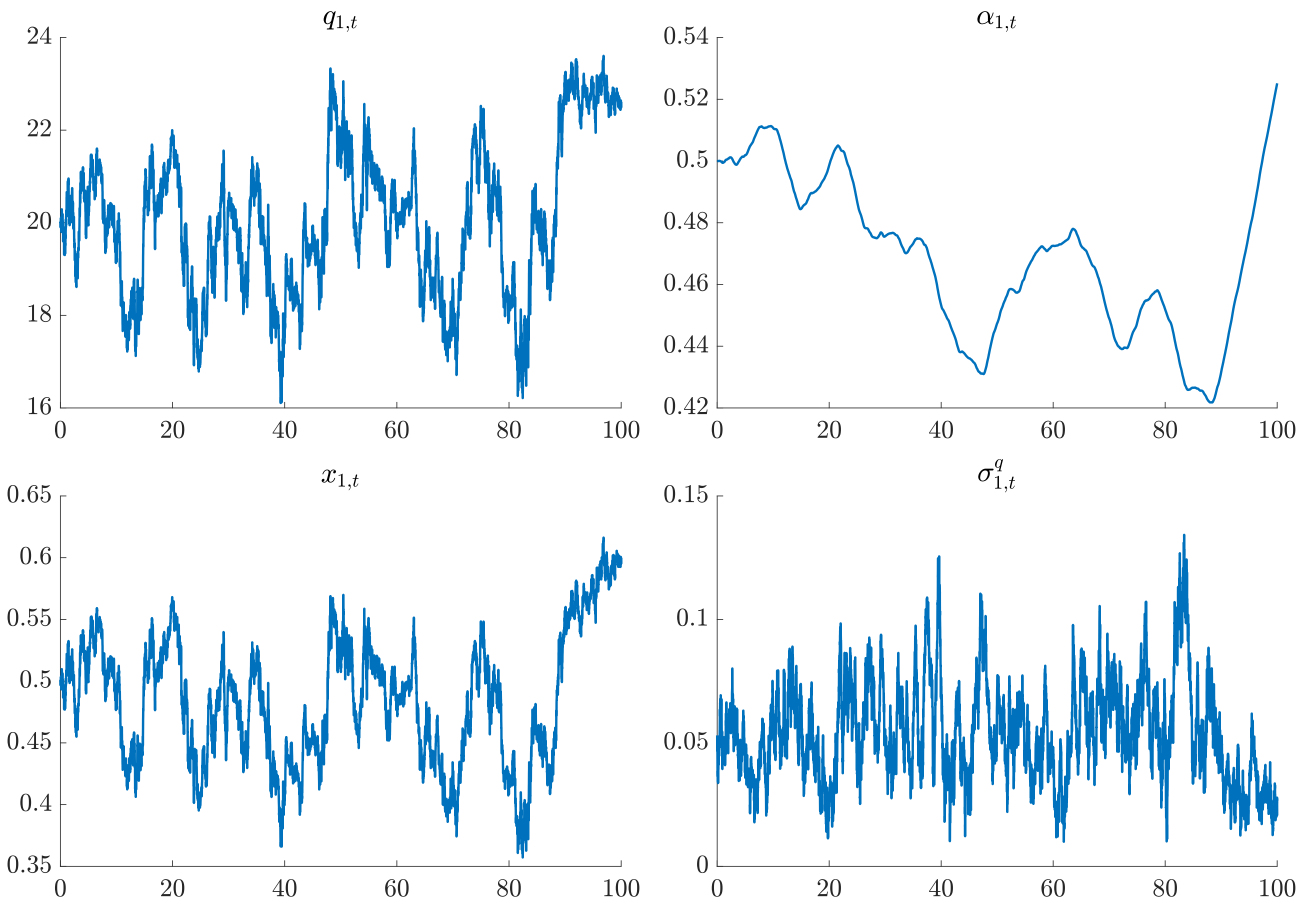


Figure E.2: Stochastic simulation from a 2-country model.

Notes. The panels display q_1, α_1, x_1, and σ_1^q from a 100-year simulation of an $N = 2$ economy. The sunspot shock exposure matrix is set so that there is one relevant sunspot shock (i.e., $Z_{1,t}$). In particular, we set

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

which has null-space $v^* = (1,1)'/\sqrt{2}$. The volatility process ψ_t is then set as follows. Let Ψ_t be a Feller square-root process

$$d\Psi_t = -\rho_\Psi(\bar{\Psi} - \Psi_t)dt + \sigma_\Psi\sqrt{\Psi_t}dZ_{1,t}.$$

Along the simulation, define the object

$$\iota_t := \mathbf{1}\Big\{\max_n(q_{n,t}/q^* - 1) > 0.2\Big\} + \mathbf{1}\Big\{\min_n(1 - q_{n,t}/q^*) > 0.2\Big\} + \mathbf{1}\Big\{\delta \geq -0.001 + \lambda\delta^{-1}\sum_{n=1}^{2}\Phi_{n,t}(1-\alpha_{n,t})\Big\}$$

Put

$$\psi_t = \begin{cases} \sqrt{\Psi_t}, & \text{if } \iota_t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Parameters: $\delta = 0.05$, $g = 0.02$, $\lambda = 0.0067$, $\phi = 0.75$, $\nu = 0$, $\hat{\nu} = 0$, $\bar{\Psi} = 1$, $\sigma_\Psi = -1$, $\rho_\Psi = 1.5$.

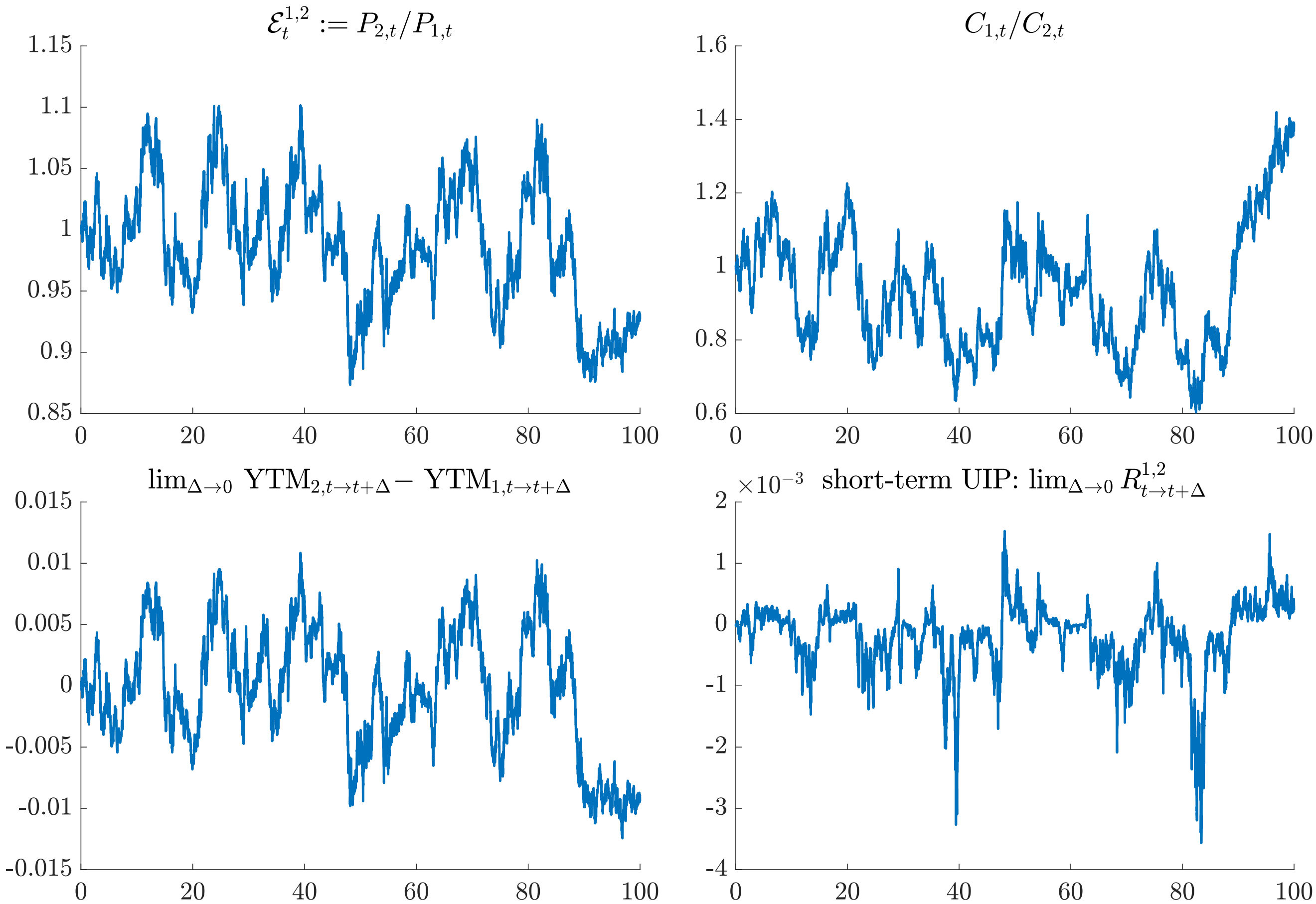


Figure E.3: Stochastic simulation from a 2-country model.

Notes. The panels display a path of the exchange rate $\mathcal{E} = \mathcal{E}^{1,2} = P_2/P_1$, the consumption ratio C_1/C_2, the difference in short-term bond yields $\text{YTM}_2 - \text{YTM}_1$, and the short-term UIP deviation $\text{YTM}_2 - \text{YTM}_1 + \mathbb{E}[d \log \mathcal{E}]$ from a 100-year simulation of an $N = 2$ economy. The specification and parameters are the same as in Figure E.2.

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