# Segmentation and Beliefs:
# A Theory of Self-Fulfilling Idiosyncratic Risk*

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**Abstract**

We study a multi-location model with financial market segmentation that permits self-fulfilling fluctuations. Such fluctuations are necessarily idiosyncratic, but their volatility varies systematically with an aggregate latent factor. We thus provide a coordination-based microfoundation for time-varying idiosyncratic risk. A key assumption of our analysis is that cash flow growth rates (e.g., firm profit growth, asset dividend growth, or country output growth) rise with valuations. We consider three applications: (i) firm dynamics and their risk factor structure; (ii) law of one price violations in finance; and (iii) exchange rate disconnect in international macroeconomics.

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Capital markets are, at least somewhat, segmented. Not all capital market participants broadly diversify across all markets. And the marginal investor in some markets trades exclusively in them. Many empirical studies have documented examples of segmentation in various contexts.[1]

In this paper, we uncover a novel aspect of capital market segmentation: self-fulfilling asset price fluctuations. Our theory is based on market segmentation along with a feedback effect between financial markets and the real economy. Broadly speaking, our paper sheds light on the following questions. Why are asset prices so volatile, in excess of cash flows and other "fundamentals"? To this classic puzzle, we add that excess volatility is particularly likely to emerge when specialists, rather than broadly diversified investors, set prices. More specific to our particular framework, what is the source of idiosyncratic uncertainty? And why does idiosyncratic uncertainty vary over time systematically? Our model predicts that self-fulfilling fluctuations are necessarily idiosyncratic, but their volatility may follow a systematic factor structure.

In our model, there are a set of $N$ abstract "locations" each of which receives its own endowment. Depending on the application, one can think of locations as firms, industries, sectors, financial markets, or countries. Each location $i$ has an equity market, segmented from the other equity markets $j \neq i$, which trades claims on its local endowment stream. It is this equity valuation that will be subject to multiple self-fulfilling equilibria. The multiplicity comes about because of an assumption, our twist relative to the existing literature, that connects each location's endowment growth rate to its valuation. Specifically, the growth rate of an endowment is assumed to be positively related to its endogenous valuation (price-dividend ratio). We first explain the purpose of this assumption, along with some alternatives, and then we elaborate on the role of market segmentation.

**Growth-valuation link and alternatives.** Without a growth-valuation link, asset prices are *uniquely* determined by their fundamental values because any other price is associated with a violation of transversality. An asset with a price above fundamental value features a low dividend-price ratio. Prices must continuously rise without bound to satisfy investors and justify the high price. Long-lived investors understand this instability, so a unique fundamental value prevails, and self-fulfilling volatility is not possible.

[1] For example, there is the well known "home bias" among international asset holdings (French and Poterba, 1991). In mortgage-backed securities markets, Gabaix et al. (2007) find that a stochastic discount factor that is based on MBS-specific risk better explains the price of prepayment risk rather than one based on aggregate wealth. Describing the market for catastrophe insurance, Froot and O'Connell (1999) find that most corporations and households self-retain exposures to catastrophic risk. Evidence of segmentation has also been found in Treasury markets (Hu et al., 2013) and convertible bond markets (Mitchell et al., 2007).

With a growth-valuation link, asset prices still obey fundamental values (our model is bubble-free), but *many* fundamental values can be sustained in equilibrium. Intuitively, if cash flows grow faster when prices rise, investors who trade a richly priced asset today will tolerate future price declines because high future cash flow growth is enough to satisfy their required returns. This growth-valuation link acts as a *stabilizing force* that keeps price-dividend ratios stationary. Transversality is generically satisfied, which opens the door to a multiplicity of valuations and self-fulfilling price volatility.

How should one understand our critical growth-valuation link? Our baseline interpretation comes from the expansive literature on feedback effects between asset prices and corporate decisions (see the survey in Bond et al., 2012). When managers can learn information from stock or bond prices, they incorporate this data into their capital expenditure decisions. The feedback between prices and investment creates a link between publicly available prices and the cash flows underlying those prices. But as we discuss in the paper, all we need is some endogenous force that keeps valuations stationary if they ever deviate from steady state.

To avoid relying on any particular endogeneity, our Internet Appendix provides three additional examples that also support self-fulfilling fluctuations. The first replaces an actual growth-valuation link with a perceived growth-valuation link: investors' optimism rises with asset prices, as in models of extrapolative beliefs (e.g., Barberis et al., 2015). The second analyzes investment under "debt overhang" (e.g., Hennessy, 2004): high prices reduce the debt overhang problem and boost investment, which raises growth rates. The third alternative example comes from the endogenous growth literature on "creative destruction" (e.g., Aghion and Howitt, 1992): high prices of incumbent firms discourage new firm entry and shrink the obsolescence rate of current products.

**Role of market segmentation.** Given the discussion so far, readers may think that a growth-valuation link alone permits self-fulfilling volatility that has nothing to do with market segmentation. In fact, the presence of many segmented markets is essential for self-fulfilling volatility, a point we now address.

First, consider a single-location economy with cash flow $y_t$. The price-dividend ratio $q_t$ in this economy cannot feature self-fulfilling volatility, even with an assumed link between $q_t$ and the growth rate of $y_t$. Why? Suppose $q_t$ were to decline for reasons unrelated to fundamental cash flows or discount rates. Having less wealth after the shock, investors will want to cut their consumption $c_t$. But in a closed economy, $c_t = y_t$. In other words, there is no mechanism to absorb the desired savings by investors, which must be zero in aggregate.

Similarly, in a multiple-location economy evolving under autarky, each location behaves like its own closed economy. Building off the logic from a single-location economy, self-fulfilling volatility cannot exist if all markets are completely segmented.

But if there exists an integrated bond market, then self-fulfilling volatility is possible in segmented asset markets. To understand the mechanism, imagine a simplified version of our model with just two markets. One group of investors ($A$-types) only trades in market $A$, whereas the second group ($B$-types) only trades in market $B$. Suppose the price of asset $A$ declines for reasons unrelated to cash flows or discount rates. Having less wealth after the shock, $A$-types will want to cut consumption and save a portion of asset $A$'s cash flows in the bond market. By bond market clearing, $B$-types must borrow this amount and consume more than asset $B$'s cash flows. This consumption plan is only optimal, however, if $B$-types' wealth has increased, which requires markets $A$ and $B$ to experience equal and opposite changes in value. Thus, the self-fulfilling volatility in our setting is always characterized by *redistribution* of wealth between markets. In this sense, our self-fulfilling fluctuations are *idiosyncratic shocks*.

A novel prediction is that asset booms are less likely to be synchronized global phenomena and more likely to be found in individual sectors and geographic locations (Brunnermeier and Schnabel, 2015). Instead of being in sync, asset boom-bust cycles should co-move negatively: a crash in one asset market necessarily coincides with a boom in another. Coincident with this wealth redistribution are a type of capital flows—the rising market borrows from the falling market—that are sometimes connected to boom-bust cycles (Caballero et al., 2008).

**Applications.** We consider three applications of our framework in Section 4. First, we interpret our locations as firms, and we interpret the agents in the model as corporate insiders that hold concentrated positions in the firm. With this interpretation, our model produces firm-level idiosyncratic stock returns whose volatility has a factor structure. In particular, firm-specific risk rises and falls systematically. Because corporate insiders hold undiversified exposures to their own stocks, firm-specific shocks command a risk premium, whose magnitude is a function of the aggregate idiosyncratic volatility factor. As we discuss in Section 4.1, these patterns are supported by the empirical finance literature on firm dynamics (Hopenhayn, 1992; Sutton, 1997; Luttmer, 2007; Gabaix, 2009) and firm-specific stock returns (Campbell et al., 2001; Herskovic et al., 2016).

Second, we interpret our locations as distinct markets for an identical asset. For this application to make sense, asset cash flows must be identical, so we replace our baseline growth-valuation link with a link between valuations and beliefs, as mentioned above. In this context, self-fulfilling volatility causes the prices of these assets to diverge, resulting in law of one price violations. Furthermore, arbitrage trades that attempt to buy underpriced assets and sell overpriced assets are not risk-free; self-fulfilling volatility implies short-run price risk in such trades. As we discuss in Section 4.2, several such arbitrage trades have been documented empirically and linked to market segmentation and slow-moving capital (Chen and Knez, 1995; Lamont and Thaler, 2003; Krishnamurthy, 2002; Ofek et al., 2004; Mitchell et al., 2007; Hu et al., 2013; Du et al., 2018, 2019; Fleckenstein and Longstaff, 2018; Makarov and Schoar, 2020).

Third, we extend the model to include "non-tradable" consumption goods and interpret our locations as countries. Self-fulfilling volatility in asset prices now spills over into real exchange rates. This volatility is in excess of fundamentals, it creates unshared risks, and it garners a risk premium, all of which help resolve various exchange rate puzzles (e.g., the PPP, Backus-Smith, and UIP puzzles). Section 4.3 discusses these puzzles in more detail, along with a growing international macro literature that embraces market incompleteness in pursuit of resolutions (Gabaix and Maggiori, 2015; Lustig and Verdelhan, 2019; Itskhoki and Mukhin, 2021).

**Contributions to the literature on multiple equilibria.** Our construction of self-fulfilling equilibria shares a similar flavor to seminal studies that build sunspot shocks around a stable steady state. We differ from this literature in some of the assumptions we adopt—we require neither overlapping generations (Azariadis, 1981; Cass and Shell, 1983; Farmer and Woodford, 1997) nor aggregate increasing returns (Farmer and Benhabib, 1994) to induce stability. Instead, we provide several new examples of "stabilizing forces." Our equilibrium construction is also more general in permitting an arbitrary numbers of markets, arbitrary fundamental shocks, and a broad class of self-fulfilling shocks.

One can think of our fluctuations as a microfoundation for volatility stemming from noise traders (De Long et al., 1990a,b, 1991), which has proved useful as a device in asset pricing (Vayanos and Vila, 2021) and international macro (Itskhoki and Mukhin, 2021). This analogy helps explain why our self-fulfilling volatility is able to resolve some arbitrage and exchange-rate puzzles.

A key feature of our analysis is that self-fulfilling fluctuations cannot be aggregate phenomena. This result echoes Loewenstein and Willard (2006), who show that noise-trader volatility in De Long et al. (1990a) cannot survive the endogeneity of the interest rate coming from bond market clearing. This result also distinguishes our mechanism from several other studies that build multiplicity through collateral constraints or other financing frictions (Krishnamurthy, 2003; Benhabib and Wang, 2013; Miao and Wang, 2018; Schmitt-Grohé and Uribe, 2021), which continue to operate in single-location, closed-economy settings.

Instead, our model features wealth redistribution through asset price fluctuations, as in the OLG model of Gârleanu and Panageas (2020) and the limited enforcement model of Zentefis (2022). Like those models, our multiplicity arises when there are multiple traded assets and some segmentation between them. Our contribution is to provide a much more general analysis, with several new applications connecting volatility to puzzles in the literature.

**Outline.** The remainder of the paper proceeds as follows. Section 1 describes the model. Section 2 analyzes some benchmark cases that cannot have multiplicity. Section 3 contains our main results on self-fulfilling volatility, focusing on the redistributive nature of self-fulfilling shocks, the role of the growth-valuation link, and the role of our financial market structure. Section 4 studies some applications of the model. That section also contains lengthy discussions of the existing literature in the context of each application.

# 1 Model

An endowment economy is set in continuous time that is indexed by $t \geq 0$.

**Endowments.** There are $N$ "locations" in the economy. Each location can stand for a firm, a sector, an industry, a country, or a distinct financial market. Each location $n$ receives an endowment stream $y_{n,t}$, with the aggregate endowment denoted by $Y_t := \sum_{n=1}^N y_{n,t}$. The endowment of location $n$ follows

$$dy_{n,t} = y_{n,t}\Big[g_{n,t}dt + \nu dB_t + \hat{\nu} d\hat{B}_{n,t} - \hat{\nu}\sum_{i=1}^N \frac{y_{i,t}}{Y_t} d\hat{B}_{i,t}\Big], \tag{1}$$

where $(B, \hat{B}_1, \dots, \hat{B}_N)$ is an $(N+1)$-dimensional standard Brownian motion. We think of $B$ as the aggregate fundamental shock and $\hat{B} := (\hat{B}_n)_{n=1}^N$ as location-specific fundamental shocks. For simplicity, each location has symmetric shock exposures $\nu$ and $\hat{\nu}$. Our results do not rely on the presence of fundamental shocks, and we could very well set $\nu = \hat{\nu} = 0$. We leave the local expected growth rate $g_n$ arbitrary for now and discuss this growth rate in more detail below. Summing across $n$ in Eq. (1), the aggregate endowment follows

$$dY_t = Y_t[g_t dt + \nu dB_t]. \tag{2}$$

We have purposefully specified location-specific shock exposures in Eq. (1) in order that the aggregate volatility is the constant $\nu$ in Eq. (2).

**Financial Markets.** Each location offers a single asset in positive net supply that is a claim to its local endowment $y_{n,t}$—we refer to this as the local equity market. The equilibrium equity price in location $n$ is $q_{n,t} y_{n,t}$, where $q_{n,t}$ is the price-dividend ratio. In addition to these $N$ distinct equity markets, there is a risk-free bond in zero net supply that offers equilibrium interest rate $r_t$. Finally, there is an integrated futures market for trading claims on the fundamental shocks $(B, \hat{B}_1, \ldots, \hat{B}_N)$, with each future in zero net supply. Allowing these futures markets is not critical, and will be relaxed in some examples below, but affords theoretical clarity to our results on multiplicity, in the sense that we isolate the minimal needed deviation from perfect markets.

A different representative agent lives in each location. Each agent can invest *only* in his or her local asset market, the short-term bond market, and the futures markets. Hence, local equity markets are segmented, but the bond and futures markets are integrated. The bond market is critical, because it allows consumption goods to be traded across locations. The fact that equity markets are segmented implies that each location has a potentially different stochastic discount factor (state-price density) $\xi_{n,t}$.

**Budgets and Constraints.** Based on the assumptions so far, the financial wealth $w_{n,t}$ representative agent in location $n$ evolves as

$$
\begin{aligned}
dw_{n,t} = (w_{n,t} r_t - c_{n,t})dt + \vartheta_{n,t}(\eta_t dt + dB_t) + \hat{\vartheta}_{n,t} \cdot (\hat{\eta}_t dt + d\hat{B}_t) \\
+ \theta_{n,t}\Big(\frac{1}{q_{n,t}} dt + \frac{d(q_{n,t} y_{n,t})}{q_{n,t} y_{n,t}} - r_t dt\Big), \quad w_{n,0} = q_{n,0} y_{n,0}.
\end{aligned} \tag{3}
$$

The terms $\vartheta_{n,t}$ and $\hat{\vartheta}_{n,t}$ represent positions in the futures markets, which have unit exposure to the shocks $(B, \hat{B})$ and earn those shocks' market prices of risk $(\eta, \hat{\eta})$, to be determined in equilibrium. The term $\theta_{n,t}$ is the agent's local equity market position, so $w_{n,t} - \theta_{n,t}$ represents the amount of saving (borrowing, if negative) in the bond market. The initial condition $w_{n,0} = q_{n,0} y_{n,0}$ says that the agent's initial endowment is a single share of the local equity, although this does not necessarily pin down their initial wealth, as the price $q_{n,0}$ is endogenous. In addition to Eq. (3), the agent must obey the solvency constraint $w_{n,t} \geq 0$ (this is the natural borrowing limit) and the No-Ponzi condition

$$
\lim_{T \to \infty} \xi_{n,T}(w_{n,T} - \theta_{n,T}) = 0. \tag{4}
$$

The No-Ponzi condition prohibits asymptotic indebtedness.

**Preferences.** Agents have infinite lives, CRRA utility with risk aversion $\rho$, time discount rate $\delta > 0$, and rational expectations. Mathematically, preferences are represented by

$$\mathbb{E}_0\Big[\int_0^\infty e^{-\delta t}\frac{c_{n,t}^{1-\rho}-1}{1-\rho}dt\Big]. \tag{5}$$

We will assume $\rho \geq 1$, in order to simplify some of our theoretical arguments and proofs; this is also the empirically-relevant case. The limiting case $\rho = 1$ corresponds to logarithmic utility, which we will use to illustrate many results.

**Market Clearing.** Clearing of the goods and bond markets is standard: $\sum_{n=1}^N c_{n,t} = Y_t$ and $\sum_{n=1}^N (w_{n,t} - \theta_{n,t}) = 0$. In addition, all the futures markets need to clear, so $\sum_{n=1}^N \vartheta_{n,t} = 0$ and $\sum_{n=1}^N \hat{\vartheta}_{n,t} = 0$. Local equity market clearing is $\theta_{n,t} = q_{n,t} y_{n,t}$ for each $n$. Finally, combining the bond and equity market clearing conditions leads to the convenient aggregate wealth constraint $\sum_{n=1}^N w_{n,t} = \sum_{n=1}^N q_{n,t} y_{n,t} = Q_t Y_t$, where $Q_t$ is the aggregate price-dividend ratio.

**Growth Rates.** To obtain our interesting multiplicity results, we will model a type of endogeneity in fundamental growth rates. We assume local growth rates take the form

$$g_{n,t} = \Gamma(q_{n,t}), \quad \forall n, \tag{6}$$

for some common bounded function $\Gamma$. The nature of the function $\Gamma$ will play a key role in the nature of equilibrium. Eq. (6) is a reduced-form representation of a micro-founded link between dividend growth and asset prices. One microfoundation of this link is that asset prices carry payoff-relevant information. Corporate managers filter this information from stock prices and update their investment decisions accordingly (Chen et al., 2007; Bakke and Whited, 2010; Goldstein and Yang, 2017; Bond et al., 2012). Under this interpretation, $\Gamma$ should be an increasing function. In some cases, we assume $\Gamma(\cdot)$ is a constant, shutting down the growth-valuation link as a benchmark. Internet Appendix C provides three alternatives to Eq. (6) that also generate the possibility of non-fundamental volatility—we discuss these alternatives in Section 3.7 in more detail.[2]

[2] Internet Appendix C.1 assumes that, rather than a real growth-valuation link, there is a *perceived* growth-valuation link, i.e., beliefs are influenced by asset valuations. Internet Appendix C.2 analyzes under-investment, of the type induced by "debt overhang" (e.g., Hennessy, 2004; DeMarzo et al., 2012). Internet Appendix C.3 analyzes an overlapping generations economy with "creative destruction" (e.g., Gârleanu and Panageas, 2020).

Collecting these points, and adding one technical inequality to ensure equilibrium existence, we make the following standing assumption:

**Assumption 1.** *Eq. (6) holds with $\Gamma(\cdot)$ continuously differentiable, bounded, and non-decreasing. In addition, $(\rho - 1)\lim_{q\to\infty}\Gamma(q) > \frac{1}{2}\rho(\rho-1)\nu^2 - \delta$.*

**Extrinsic Shocks.** To introduce and allow the possibility of non-fundamental volatility, conjecture that the price-dividend ratio of each location's asset follows a stochastic process of the form

$$dq_{n,t} = q_{n,t}\Big[\mu^q_{n,t}dt + \varsigma^q_{n,t}dB_t + \hat{\varsigma}^q_{n,t}\cdot d\hat{B}_t + \tilde{\sigma}^q_{n,t}\cdot d\tilde{Z}_t\Big], \tag{7}$$

where $\tilde{Z}$ is an $N$-dimensional Brownian motion, independent from $B$ and $\hat{B}$. The shock $\tilde{Z}_t$ is *extrinsic*, and it is the source of self-fulfilling fluctuations, if any exist.

Economically, the extrinsic $\tilde{Z}$ shocks arise from sources that we do not explicitly model—they are sunspot shocks. In all papers with sunspot shocks, a common question is "what is the sunspot?" We do not take any stand on this, but there are several possibilities explored in the literature. One popular candidate is investor sentiment or signals that coordinate beliefs (Benhabib et al., 2015); other candidates highlighted by the literature are shocks with vanishingly small impacts on fundamentals so that they are effectively extrinsic but still retain a coordination role (Manuelli and Peck, 1992).

For ease of exposition and interpretation, we slightly restrict the shock exposure vector $\tilde{\sigma}^q_{n,t}$ as follows. Suppose there exists a time-dependent scalar $\sigma^q_{n,t} \geq 0$ and a time-independent row vector $M_n$ such that

$$\tilde{\sigma}^q_{n,t} = \sigma^q_{n,t}M_n. \tag{8}$$

Without loss of generality, suppose $M_n$ has unit norm. Restriction (8) is equivalent to saying that non-fundamental fluctuations have a constant correlation across locations. Indeed, $\mathrm{corr}_t[\tilde{\sigma}^q_{i,t}\cdot d\tilde{Z}_t, \tilde{\sigma}^q_{j,t}\cdot d\tilde{Z}_t] = (\sigma^q_{i,t}M_i \cdot M_j\sigma^q_{j,t})/(\sigma^q_{i,t}\sigma^q_{j,t}) = M_i \cdot M_j$. This constant-correlation restriction on sunspot shocks is sufficiently rich for our purposes. Despite the fact that $M$ is constant over time, it is important to remember that it is still endogenous and determined in equilibrium.

The expositional value of restriction (8) is that we can now refer to the scalar $\sigma^q_{n,t}$ as the *self-fulfilling volatility* of location $n$. In particular, define

$$Z_t := M\tilde{Z}_t, \tag{9}$$

where $M := (M_1', M_2', \dots, M_N')'$ captures cross-sectional correlations. Then, $Z_{n,t} = M_n \tilde{Z}_t$ is a one-dimensional Brownian motion, and the self-fulfilling shock to asset $n$ is $\sigma_{n,t}^q dZ_{n,t}$. If $\sigma_{n,t}^q > 0$ for some $n$, we will say that the economy exhibits self-fulfilling volatility; if $\sigma_{n,t}^q = 0$ for all $n$, we will say self-fulfilling volatility does not exist.

**No-Bubble Assumption.** As a consequence of the No-Ponzi conditions (4) and individual agents' transversality condition $\lim_{T\to\infty} \mathbb{E}_t[\xi_{n,T} w_{n,T}] = 0$, it is possible to show that $\lim_{T\to\infty} \xi_{n,T} q_{n,T} y_{n,T} = 0$ holds in any equilibrium. This is enough for our purposes, but we impose the following slightly stronger "no-bubble" condition for theoretical clarity.

**Condition 1.** For each $n$, it holds that $\lim_{T\to\infty} \mathbb{E}_t[\xi_{n,T} q_{n,T} y_{n,T}] = 0$.

Because of Condition 1, equity prices equal present values of future dividends. Self-fulfilling volatility in our model is thus consistent with classical no-bubble theorems (e.g., Santos and Woodford, 1997; Loewenstein and Willard, 2000) that give conditions under which bubbles are not possible.

**Equilibrium.** This completes the description of the model. An *equilibrium* is a set of adapted processes $(y_{n,t}, c_{n,t}, w_{n,t}, q_{n,t}, \xi_{n,t}, \theta_{n,t}, \vartheta_{n,t}, \hat{\vartheta}_{n,t})_{t\geq 0}$ for $1 \leq n \leq N$ and $(r_t, \eta_t, \hat{\eta}_t)_{t\geq 0}$, adapted to the augmented filtration generated by $(B, \hat{B}, \tilde{Z})$, such that: agents maximize (5) subject to their budget constraint (3), their No-Ponzi condition (4), and their solvency constraint $w_{n,t} \geq 0$; Eqs. (1), (6), (7), (8), and (9) all hold; all markets clear; and Condition 1 holds. In Appendix A, we derive the complete set of conditions characterizing equilibrium that we will use going forward. In expositing our results below, we will bring forth and explain any critical equations, so it will not be necessary for the reader to consult Appendix A unless a detailed derivation is desired.

**Endowment and consumption shares.** Because of the scalability properties of our model, we will repeatedly make use of the endowment and consumption shares to characterize equilibrium:

$$\alpha_{n,t} := \frac{y_{n,t}}{Y_t} \quad \text{and} \quad x_{n,t} := \frac{c_{n,t}}{Y_t}. \tag{10}$$

The dynamics of all stationary variables can be described without reference to $Y_t$, once we know $(\alpha_{n,t}, x_{n,t})_{n=1}^N$.

## 2 Benchmarks without self-fulfilling volatility

In order to understand the necessary conditions for our results, we begin with some benchmark cases that cannot support self-fulfilling volatility. These benchmarks will illustrate that self-fulfilling volatility can only arise under three conditions: (1) there must be some endogeneity in growth rates; (2) our economy must have multiple locations, hence multiple endowments; and (3) there must be an integrated bond market, hence goods trading across the locations.

**Benchmark 1: No Endogeneity.** We begin without the endogeneity of growth rates and beliefs captured by $\Gamma(\cdot)$ in Eq. (6). In other words, all endowments grow at the common constant rate $g$. It turns out that the equilibrium is unique and not volatile in this case.

The general intuition is the equilibrium dynamical system for asset prices is "unstable." Said differently, if asset valuations ever deviated from their "steady-state" value, they would necessarily explode to infinity, in violation of a transversality condition. With unstable dynamics, asset valuations must always remain at the steady-state, irresponsive to extrinsic shocks.

To illustrate the point transparently in a simple way, consider a special case of the economy with no fundamental risks, $\nu = \hat{\nu} = 0$, and with a deterministic equilibrium. In such an equilibrium, each location's asset must satisfy the following standard Euler equation (i.e., its return must equal the interest rate):

$$\frac{\dot{q}_{n,t}}{q_{n,t}} + g + \frac{1}{q_{n,t}} = r_t. \tag{11}$$

Furthermore, since individual consumption paths are deterministic, the equilibrium interest rate is solely determined by time-discounting and economic growth: $r_t = \delta + \rho g$. Substituting this expression into Eq. (11) gives for each location

$$\dot{q}_{n,t} = -1 + [\delta + (\rho - 1)g]q_{n,t}. \tag{12}$$

By Assumption 1, the term in square brackets is positive, so Eq. (12) represents an unstable dynamical system with the unique steady state $q^* = \frac{1}{\delta+(\rho-1)g} > 0$. As we will show in Section 3, the precise role of growth endogeneity will be to render the dynamical system stable rather than unstable.

The dynamical instability in Eq. (12) implies that $q_{n,t} = q^*$ at all times. Indeed, if a price-dividend ratio is below (above) $q^*$, it drifts downwards (upwards) at a pace that accelerates over time. This explosive behavior ultimately violates either free disposal (which requires $q_{n,t} \geq 0$) or the no-bubble condition in Condition 1 (which requires $q_{n,t}$ not explode upwards), so it cannot be an equilibrium. The general proof allows $\nu > 0$ and $\hat{\nu} > 0$ and an arbitrary stochastic equilibrium, which requires a more involved analysis, but the core intuition is this dynamical instability.

**Lemma 1.** *Assume constant growth, $\Gamma(q) \equiv g$. Then, no self-fulfilling volatility can exist.*[3]

*Proof.* See Appendix B.2. □

**Benchmark 2: One Location.** Next, consider a single-location economy, $N = 1$. With the possibility of endogenous growth, can self-fulfilling volatility exist? The answer turns out to be no, because cash flow growth also shows up in discount rates, which then offsets the effects of endogenous cash flows on valuation.

To see this, note that the representative agent consumes aggregate output $Y_t$, and so the state-price density $\xi_t = e^{-\delta t} Y_t^{-\rho}$ evolves as

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \eta_t dB_t,$$

where $r_t = \delta + \rho\Gamma(Q_t) - \frac{1}{2}\rho(\rho+1)\nu^2$ is the short-term interest rate and $\eta_t = \rho\nu$ is the market price of aggregate risk. Right away, one sees that the growth rate $\Gamma(Q_t)$ shows up in the interest rate, so the endogeneity in $\xi_t$ will offset the endogeneity of $Y_t$ in the present-value equation

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty \frac{\xi_s}{\xi_t}\frac{Y_s}{Y_t} ds\Big].$$

Indeed, the Euler equation is $\mu_t^Q + g_t + \frac{1}{Q_t} + \nu\varsigma_t^Q = r_t + (\nu + \varsigma_t^Q)\eta_t$, which after some algebra says

$$\mu^Q = \delta - \frac{1}{2}\rho(\rho-1)\nu^2 + (\rho-1)\Gamma - \frac{1}{Q} + (\rho-1)\nu\varsigma^Q. \tag{13}$$

Eq. (13) represents an unstable dynamical system. This instability can seen clearly by setting $\varsigma^Q = 0$ so that

$$\mu^Q = f(Q) := \delta - \frac{1}{2}\rho(\rho-1)\nu^2 + (\rho-1)\Gamma(Q) - \frac{1}{Q}.$$

Now, $f(Q) = 0$ has at least one root $Q^*$, because $f(0+) = -\infty$ and $f(+\infty) > 0$ (by Assumption 1). This root is unique because $f'(Q) > 0$, given $\rho \geq 1$. Hence, if $Q_t \neq Q^*$

[3] In fact, we show equilibrium is unique, with $q_{n,t} = q^* := (\delta + (\rho-1)g - \frac{\rho(\rho-1)}{2}\nu^2)^{-1}$ for each $n$.

ever occurred, the drift $\mu_t^Q$ would cause $Q_t$ to diverge further and further away from $Q^*$. This line of argument suggests that equilibrium should always have $Q_t = Q^*$.[4]

**Lemma 2.** *Suppose $N = 1$. Suppose $(\rho - 1)\Gamma(0) > \frac{1}{2}\rho(\rho - 1)\nu^2 - \delta$. Then, no self-fulfilling volatility can exist.*

*Proof.* See Appendix B.3. □

**Benchmark 3: Autarky.** Due to Lemma 2, we know that self-fulfilling volatility requires multiple locations. But these locations cannot be too segmented; in particular, sunspot volatility cannot survive in autarky. This argument is quite simple: each location is a closed economy in autarky, and so the basic reasoning of Lemma 2 applies.

**Lemma 3.** *Assume autarky, in the sense that agent $n$ consumes his endowment $c_{n,t} = y_{n,t}$. In addition, suppose $(\rho - 1)\Gamma(0) > \frac{1}{2}\rho(\rho - 1)\nu^2 - \delta$. Then, no self-fulfilling volatility can exist.*

*Proof.* See Appendix B.4. □

The key takeaway from Lemma 3 is the importance of the bond market. Indeed, the bond market is precisely the mechanism through which goods trading is possible and autarky is escaped: one location can consume beyond its endowment only by borrowing from another location. Thus, our self-fulfilling fluctuations require multiple equity markets and an integrated bond market.

**Remark 1** (Small open economy)**.** Although the emergence of self-fulfilling volatility requires an open and active bond market (Lemma 3), it does not require bond market clearing. Consider a "small open economy" in which the equity market for claims to the stream $\{y_{n,t}\}_{t \geq 0}$ clears for each $n$, but the bond market does not. It turns out that such an economy will possess indeterminacy and the potential for self-fulfilling volatility. Intuitively, a "small open economy" can be seen simply as one location in a multi-location closed economy. In that sense, the bond market is open and active with the modeled location borrowing from / lending to the rest of the world.

[4] Unfortunately, it is quite challenging to prove $Q_t = Q^*$ in our general environment. The main complication is proving instability of the dynamical system while allowing $\varsigma^Q \neq 0$. To tackle this, we leverage tools from the mathematics literature on backward stochastic differential equations (BSDEs), tools which are readily applicable to our model.

# 3 Self-fulfilling volatility

In this section, we provide a broad characterization of self-fulfilling volatility in general and in more detail with a specific example. Section 3.1 demonstrates that volatility must always be redistributive across locations and illustrates redistribution with some examples. Section 3.2 proves a sufficient condition—namely stationarity of valuations and the consumption distribution—that any conjectured equilibrium must satisfy in order to permit self-fulfilling volatility; this condition serves as a useful tool in equilibrium constructions. Section 3.3 provides an explicit example, with a linear growth-valuation link, of an equilibrium with self-fulfilling volatility. Section 3.4 generalizes the example to have a nonlinear growth-valuation link. Section 3.5 elaborates on the role of the bond market and volatility-induced precautionary savings. Section 3.6 shows that complete financial markets can also support self-fulfilling price volatility, but without any interesting "real" consequences, which will be important for our applications below. Finally, Section 3.7 provides several other alternative sources of endogeneity, that might replace the growth-valuation link, which also support multiplicity.

## 3.1 Redistribution

As Lemma 2 suggests, self-fulfilling volatility is not likely to be an aggregate phenomenon. We formalize this by showing that extrinsic shocks necessarily *redistribute* wealth across markets, in a certain sense.

To see the logic most transparently, assume log utility ($\rho = 1$) for the sake of this discussion. Investors with log utility consume a fraction $\delta$ of their wealth, so the aggregate wealth-consumption (price-dividend) ratio is $Q_t = \delta^{-1}$. Bond market clearing can then be written as

$$\sum_{n=1}^{N} \alpha_{n,t} q_{n,t} = \delta^{-1}, \tag{14}$$

where $\alpha_{n,t} := y_{n,t}/Y_t$ is the location-$n$ endowment share. Because the aggregate wealth-consumption ratio is constant, if any extrinsic shocks affect $q_{n,t}$, they must be offset by extrinsic shocks to other assets. Hence, if extrinsic shocks influence prices, these shocks must redistribute wealth across markets.

Wealth redistribution is tied directly to $\mathrm{rank}(M)$. By applying Itô's formula to Eq. (14), we see that the loadings on each of the basis extrinsic shocks $d\tilde{Z}_t$ must be zero:

$$\sum_{n=1}^{N} \alpha_{n,t} q_{n,t} \sigma^{q}_{n,t} M_n = 0. \tag{15}$$

Writing Eq. (15) as a matrix equation gives

$$M' v_t = 0, \tag{16}$$

where $v_t = (\alpha_{1,t} q_{1,t} \sigma^q_{1,t}, \ldots, \alpha_{N,t} q_{N,t} \sigma^q_{N,t})'$ is the column vector of volatilities. If the matrix $M$ were full rank, the unique solution to Eq. (16) would be $v_t \equiv 0$ and there would be no self-fulfilling volatility. Therefore, if equilibrium ever features $v_t \neq 0$, then is must be that $\text{rank}(M) < N$.

In the proof of the next lemma, we generalize the above argument to non-log utility.

**Lemma 4.** *Let $N > 1$. If equilibrium features self-fulfilling volatility, i.e., $(\sigma^q_{1,t}, \ldots, \sigma^q_{N,t}) \neq 0$, then extrinsic shocks must be redistributive, in the sense that $\text{rank}(M) < N$.*

*Proof.* See Appendix B.5. □

Note that the matrix $M$ is an equilibrium object, not a parameter. This is important, because if $M$ were a parameter, our argument would imply that self-fulfilling volatility could only emerge in a zero-measure subset of our parameter space (indeed, almost every $N \times N$ matrix is full rank). Instead, Lemma 4 is saying that all equilibria with self-fulfilling volatility can be characterized by coordination on a number of shocks which is strictly fewer than the number of locations.

With this in mind, consider the following two examples of $M$ that are consistent with redistribution and thus could emerge in an equilibrium with self-fulfilling volatility.

**Example 1** (Two-by-two redistribution)**.** Suppose $N = 2$ and let

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}. \tag{17}$$

This example has two locations and one source of extrinsic uncertainty. The matrix $M$ puts $Z_{1,t} = -Z_{2,t}$, which implies that the self-fulfilling price changes redistribute wealth between the two assets. As one price falls, the other rises.

**Example 2** (General redistribution)**.** This example is the $N$-dimensional counterpart to Example 1. Let $\tilde{M}$ be an $N \times N$ non-singular matrix. Suppose

$$M = \tilde{M} - \frac{1}{N} \mathbf{1}' \tilde{M} \otimes \mathbf{1}. \tag{18}$$

In this structure, each element of the matrix $\tilde{M}$ is reduced by the simple average of its columns. This operation makes the column sums of $M$ equal zero. The key consequence of this design is that $\mathbf{1}'dZ_t = \mathbf{1}'Md\tilde{Z}_t = 0$ almost-surely. Any other linear combination of $dZ_t$ does not equal 0. As a result, $\text{rank}(M) = N - 1$. In this example, self-fulfilling price changes redistribute wealth across the $N$ markets.

We can use our redistributive characterization of self-fulfilling volatility to develop several implications, particularly concerning observed boom-bust patterns in asset markets. First, the model explains that self-fulfilling booms often occur less widely and more in isolation. In our model, self-fulfilling asset booms cannot be aggregate global phenomena. Instead, asset booms must occur in a subset of countries or asset markets, which may be why "bubbles" are often found in a specific region or asset class (Brunnermeier and Schnabel, 2015).

Second, redistribution implies that a self-fulfilling market crash in one country or asset class could coincide with a boom in an alternative country or asset class. There is some evidence for this type of relation. For example, the 1997 Asian financial crisis coincided with the start of a large boom in the US stock market, primarily in technology stocks. Also, the 2000-02 timing of the US stock market crash matched the run up of the US housing market boom. And finally, the 2006-07 housing market downturn coincided with a boom in commodities markets, mainly in oil, as discussed more formally in Caballero et al. (2008). The authors interpret these facts as a migration of a bubble due to "global imbalances." But the model here demonstrates that such a migration could also take place even without bubbles.

### 3.2 Stationarity as a sufficient condition for multiplicity

So far, we have characterized the redistributive nature of self-fulfilling volatility, if it exists. But when does such volatility exist? Before constructing specific examples, it will be helpful to identify, as a diagnostic tool, two sufficient conditions for existence of an equilibrium with self-fulfilling volatility: (i) asset prices are positive and non-explosive, and (ii) all consumers survive in the long run. These two requirements together ensure that there is free disposal of assets (i.e., no negative asset prices) as well as no Ponzi schemes (i.e., transversality holds). An interpretation of these requirements is that asset valuations and the wealth distribution are both *stationary*. In our examples below, we will translate these stationarity requirements into conditions on growth rates. For now, we summarize our discussion with the following theorem, which provides the sufficient conditions for self-fulfilling volatility in the case of log utility ($\rho = 1$). While it may be possible to extend this theorem to $\rho \neq 1$, it is difficult and beyond the scope of this paper, especially considering our examples that follow are all constructed under log utility.[5]

**Theorem 1.** *Let $\rho = 1$. Let $M$ be any $N \times N$ matrix (with rows of unit norm) having rank$(M) < N$, and let vector $v^* = (v_1^*, \ldots, v_N^*)'$ be in the null-space of $M'$. Then, for some non-zero adapted scalar process $\{\psi_t\}_{t \geq 0}$, an equilibrium exists with self-fulfilling volatility satisfying $\alpha_{n,t} q_{n,t} \sigma^q_{n,t} = \psi_t v_n^*$ for all $n$, if (i) the resulting price-dividend ratios $\{(q_{n,t})_{n=1}^N\}_{t \geq 0}$ are positive, bounded processes, and (ii) $\lim_{T \to \infty} \mathbb{E}_t[e^{-\delta T} \frac{1}{x_{n,T}}] = 0$ for each $n$.*

*Proof.* See Appendix B.6. □

### 3.3 Example with an explicit construction

This section provides an explicit example of an economy with self-fulfilling volatility. In this examples, we will specialize to log utility, $\rho = 1$, which facilitates the analysis.

Suppose the linear functional form for growth rates:

$$\Gamma(q) = g + \lambda(q - \delta^{-1}), \quad \text{with} \quad \lambda > \delta^2. \tag{19}$$

(Although the unboundedness of $\Gamma$ violates Assumption 1, this is inconsequential: in the equilibria below, $q_{n,t}$ will be bounded, so that we could truncate the function $\Gamma$ at the maximal and minimal possible values of $q_{n,t}$.) The linearity in Eq. (19) is particularly convenient, because the aggregate growth rate will always be the constant $g$. The exact quantitative connection between growth rates and prices in Eq. (19) is not too extreme: with a standard discount rate of $\delta = 0.05$, growth rates must be at least 0.5% above average when valuations are 10% above average.[6] The next proposition, which provides an explicit equilibrium construction, demonstrates the existence of self-fulfilling volatility.

**Proposition 1.** *Let $\rho = 1$, and assume either $N \geq 3$ or $\hat{v} = 0$. Assume local growth rates satisfy Eq. (19). Then, an equilibrium exists with self-fulfilling volatility.*

*In particular, let $\{\psi_t\}_{t \geq 0}$ be any non-zero process satisfying the following two properties:*

*(P1) $\psi_t / \min_n \alpha_{n,t}$ and $\psi_t / \min_n x_{n,t}$ are bounded;*

[5] In the case of $\rho \neq 1$, the proof is much more challenging, because individual wealth-consumption ratios satisfy a system of backward stochastic differential equations (BSDEs), which are coupled with the assumed dynamics of asset valuations $(q_{n,t})_{n=1}^N$ and the endowment and consumption share state variables $(\alpha_{n,t}, x_{n,t})_{n=1}^N$. See Eq. (A.10) in Appendix A for the characterization of this BSDE system. Thus, to prove an analog to Theorem 1, we would need to verify the existence of a solution to this high-dimensional system of BSDEs.

[6] The steady-state valuation $q^* = \delta^{-1}$. For a $(100 \times p)$% higher valuation, the growth rate is therefore higher by $\Gamma((1+p)q^*) - \Gamma(q^*) = p\lambda q^* = p\lambda\delta > p\delta$. For a 10% higher valuation ($p = 0.1$) with $\delta = 0.05$, $p\delta = 0.005 = 0.5\%$. More generally, the growth-price semi-elasticity $\frac{dg_{n,t}}{d \log q_{n,t}}$ must at least be $\delta$.

*(P2)* $\psi_t$ *vanishes as* $\min_n q_{n,t}$ *approaches* $\delta(\epsilon + \lambda^{-1})$ *from above, for* $0 < \epsilon < \delta^{-2} - \lambda^{-1}$*, or as* $\max_n q_{n,t}$ *approaches* $K\delta^{-1}$ *from below, for some* $K > 1$.

*An equilibrium exists with* $\alpha_{n,t} q_{n,t} \sigma^q_{n,t} = v^*_n \psi_t$ *for all* $n$*, where* $v^*$ *is in the null-space of* $M'$.

*Proof.* See Appendix B.7. □

The sufficiently-strong dependence of growth on asset prices allows for self-fulfilling expectations of future price changes to take hold. For instance, if investors anticipate high prices, their expectations for dividend growth rates rise, which justifies the high prices and confirms the initial expectations. Conversely, if investors anticipate low prices, expected growth rates drop as well, fulfilling the starting beliefs about low prices.

Mathematically, the endogeneity of dividend growth rates acts as a force to keep valuation dynamics stable, in contrast to Lemma 1. To see this clearly, suppose $\nu = \hat{\nu} = 0$ and recall the pricing equation in a deterministic equilibrium:

$$\frac{\dot{q}_{n,t}}{q_{n,t}} + g_{n,t} + \frac{1}{q_{n,t}} = r_t. \tag{20}$$

Substitute $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$ and use the fact that $r_t = \delta + \sum_{n=1}^N \alpha_{n,t} g_{n,t} = \delta + g$. The result is

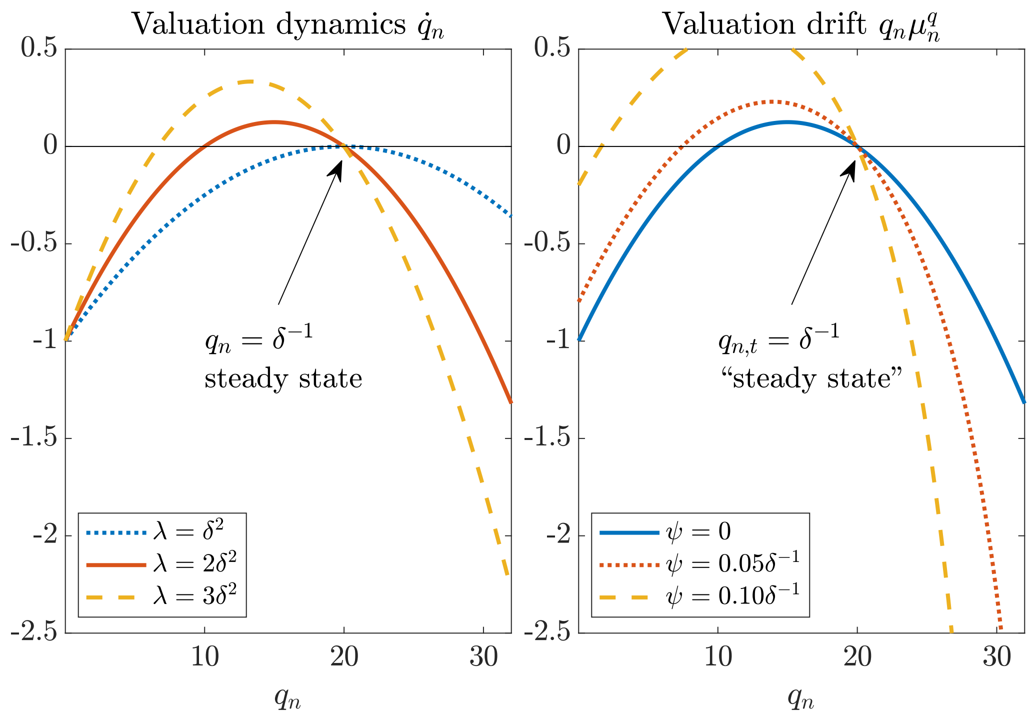
$$\dot{q}_{n,t} = -1 + \delta(1 + \lambda/\delta^2) q_{n,t} - \lambda q_{n,t}^2. \tag{21}$$

The left panel of Figure 1 provides an illustration of the dynamical system Eq. (21) for $\delta = 0.05$ and various amounts of endogeneity through $\lambda$.

The dynamical system in Eq. (21) now has two steady states. As long as $\lambda > \delta^2$, the larger of the two steady states is the relevant one (i.e., the one with $q_n = \delta^{-1}$). This larger steady state is locally stable, in the sense that $\frac{\partial \dot{q}_n}{\partial q_n}\big|_{q_n = \delta^{-1}} = \delta(1 - \lambda/\delta^2) < 0$. When the economy has this stability property, some amount of self-fulfilling volatility becomes possible. The amount of volatility is only restricted by the requirement—captured by properties (P1) and (P2) in Proposition 1—that it vanishes when the economy is "far from the steady state." (Technically, the proof is more complicated because there is no deterministic steady state when $\hat{\nu} \neq 0$.) In fact, properties (P1) and (P2) are exactly what we need to verify the stationarity requirements from Theorem 1.

More deeply, low asset valuations return to "steady state" because required returns must be satisfied by capital gains rather than fundamentals growth. In particular, Eq. (20) conveys the requirement that investors must earn $r_t$ on their investments. If $q_{n,t}$ falls a bit below its steady state, so that $g_{n,t}$ also falls, then capital gains $\dot{q}_{n,t}/q_{n,t}$ must rise to satisfy investors—this force brings valuations back up. And a reciprocal logic applies to

Figure 1: Valuation dynamics for a generic location.




*Notes.* Both panels have $\rho = 1$, $\delta = 0.05$, and $\nu = \hat{\nu} = 0$. The left panel assumes a deterministic equilibrium. The right panel has $\lambda = 2\delta^2$, $N = 2$ locations, and plots the drift from a stochastic equilibrium. The equilibrium plotted in the right panel features $M$ as in Example 1, such that $v^* = (1, 1)$ is in the null-space of $M'$. Finally, for plotting purposes, the right panel assumes $x_{n,t} = \alpha_{n,t} = 0.5$ for each $n$ (consumption and endowment shares coincide).

high asset valuations. The role of property (P2) is to keep dynamics in this stable region. If an asset valuation falls to zero, for instance, its dividend yield diverges to infinity, and so capital gains must necessarily be negative—such unstable dynamics are ruled out of $\min_n q_{n,t}$ does not fall too far.

The right panel of Figure 1 plots the expected capital gains $q_{n,t}\mu^q_{n,t}$ in a specific stochastic equilibrium example with $N = 2$ locations and the extrinsic shock correlation structure $M$ as in Example 1. The different values of $\psi$ correspond to different levels of volatility, since recall $\alpha_{n,t} q_{n,t} \sigma^q_{n,t} = \psi_t$. For $\psi = 0$ (solid line), dynamics are identical to the deterministic equilibrium, which is why the solid lines in the right- and left-hand panels coincide. For $\psi > 0$, the presence of volatility steepens the drift, because low-valuation locations have higher volatility and thus higher risk premia. Risk premia must be met by higher expected capital gains, so this force provides an increase to $\mu^q_{n,t}$ when $q_{n,t}$ is low, and vice versa when $q_{n,t}$ is high.

Indeed, the formula for the valuation drift without fundamental idiosyncratic shocks

($\hat{\nu} = 0$) and with log utility ($\rho = 1$) is

$$q_{n,t}\mu^q_{n,t} = \underbrace{-1 + \delta(1+\lambda/\delta^2)q_{n,t} - \lambda q^2_{n,t}}_{\text{deterministic component}} + \underbrace{\delta\frac{(v^*_n\psi_t)^2}{\alpha_{n,t}x_{n,t}}}_{\text{risk premium}} - \underbrace{q_{n,t}\delta^2\sum_{i=1}^{N}\frac{(v^*_i\psi_t)^2}{x_{i,t}}}_{\text{precautionary savings}} \tag{22}$$

(The general formula for $\mu^q_{n,t}$ is in Eq. (A.2) of Appendix A.) The term labeled "deterministic component" is the entire drift when $\psi_t = 0$ and is identical to $\dot{q}_{n,t}$ in Eq. (21). The term labeled "risk premium" arises because investor $n$ demands compensation for the self-fulfilling volatility in his local equity, a risk premium which must be delivered via future capital gains. We will elaborate in detail on term labeled "precautionary savings," which arises from the equilibrium interest rate, in Section 3.5 below. To see transparently the steepening effect that $\psi > 0$ has in Figure 1, simply observe that $q_{n,t}$ scales the precautionary savings term, so that tends to dominate the risk premium term when $q_{n,t}$ is high, and vice versa.

### 3.4 Generalizing the linear functional form

The reader may think the example from Proposition 1 is special in that the growth-valuation link is linear. Here, we sketch a generalization of linearity by allowing $\Gamma(q)$ to be an arbitrary increasing function satisfying $\Gamma(\delta^{-1}) = g$ and $\Gamma'(\delta^{-1}) > 0$. To facilitate the analysis, we continue to assume log utility ($\rho = 1$) and will also assume there are no exogenous shocks ($\nu = \hat{\nu} = 0$).

Consider a deterministic equilibrium. The pricing condition is still Eq. (20). But now the equilibrium interest rate is $r_t = \delta + \sum_{n=1}^{N}\alpha_{n,t}\Gamma(q_{n,t})$, which is a function of the entire cross-section of valuations. Hence, the dynamical system (21) is replaced by

$$\dot{q}_{n,t} = -1 + \delta q_{n,t} - [\Gamma(q_{n,t}) - \Gamma(\delta^{-1})]q_{n,t} + q_{n,t}\sum_{i=1}^{N}\alpha_{i,t}[\Gamma(q_{i,t}) - \Gamma(\delta^{-1})]. \tag{23}$$

The steady state of this dynamical system is $q_{n,t} = \delta^{-1}$ for all $n$. (Note that the endowment shares $(\alpha_{i,t})_{i=1}^{N}$ are evolving jointly with the valuations, as $\dot{\alpha}_{n,t} = \alpha_{n,t}[\Gamma(q_{n,t}) - g_t]$, but it will become clear below that the dynamical system for $(q_{i,t})_{i=1}^{N}$ is stable for any endowment distribution.)

Local stability properties near steady state are determined by the eigenvalues of the

$N \times N$ Jacobian matrix

$$J := \left[ \frac{\partial \dot{q}_n}{\partial q_m} \Big|_{ss} \right]_{1 \le n,m \le N} \tag{24}$$

By direct calculation,

$$\frac{\partial \dot{q}_n}{\partial q_m} \Big|_{ss} = \begin{cases} \delta - (\delta^{-1} - \alpha_n)\Gamma'(\delta^{-1}), & \text{if } m = n; \\ \alpha_m \Gamma'(\delta^{-1}), & \text{if } m \neq n. \end{cases}$$

Suppose $\Gamma'(\delta^{-1}) > \frac{\delta^2}{1-\delta}$ (note that this condition closely mirrors the condition $\lambda > \delta^2$ in Proposition 1). Then, regardless of the steady-state values of $(\alpha_n)_{n=1}^N$, the matrix $J$ is diagonally dominant with negative diagonal entries. By the Gershgorin circle theorem, all the eigenvalues of $J$ have strictly negative real parts. From standard results in dynamical systems, the dynamical system for $(q_{n,t})_{n=1}^N$ is stable. We have thus proved

**Lemma 5.** *Assume $\rho = 1$, $\nu = \hat{\nu} = 0$, and $\Gamma'(\delta^{-1}) > \frac{\delta^2}{1-\delta}$. Then, the deterministic equilibrium is locally stable.*

Given Lemma 5, it is possible to construct a stochastic equilibrium using the extrinsic shock $\tilde{Z}$ for *any* growth-valuation link that is sufficiently strong near steady state.

## 3.5 The role of the bond market

From Lemma 3, which demonstrates equilibrium uniqueness under autarky, we already knew that bond market trading would be critical to our multiplicity results. Here, we explore the bond market in more detail.

To start, consider the mechanics of self-fulfilling fluctuations through the lens of the bond market. If the valuation $q_{1,t}$ increases to an extrinsic shock $dZ_{1,t} > 0$, agent 1 will have higher future endowments via the growth-valuation link in $\Gamma(q_{1,t})$. Knowing her future endowments will be higher, it is optimal to consume now. But her local endowment $y_{1,t}$ has not changed in the short run; to consume in excess of her endowment—i.e., to consume $c_{1,t} > y_{1,t}$—she must borrow from other locations. The reverse holds for agent 2 who supplies funds to the bond market, due to a reduction in his local valuation ratio: his future endowments are lower, which incentivizes savings to smooth consumption. Without the bond market, no valuation changes could be justified.

Having understood that the bond market allows sunspot dynamics to emerge and be self-confirmed, how does the resulting volatility feed back into the bond market? A clear way to gauge this feedback is through the equilibrium interest rate, which is given by

$$r_t = \underbrace{\delta + \rho g_t - \frac{1}{2}\rho(\rho+1)\nu^2}_{\substack{\text{representative-agent} \\ \text{interest rate}}} - \underbrace{\frac{1}{2}\rho(\rho+1)\sum_{n=1}^{N} x_{n,t}(\sigma_{n,t}^c)^2}_{\substack{\text{idiosyncratic} \\ \text{precautionary savings}}} \tag{25}$$

If all locations were perfectly integrated, a representative agent would exist and the equilibrium interest rate would be $\delta + \rho g_t - \frac{1}{2}\rho(\rho+1)\nu^2$, which reflects discounting plus growth minus the precautionary savings motive due to aggregate volatility.

If locations are segmented, and in addition self-fulfilling volatility takes hold, then an additional precautionary savings term arises, namely $\frac{1}{2}\rho(\rho+1)\sum_{n=1}^{N} x_{n,t}(\sigma_{n,t}^c)^2$. In particular, $\sigma_{n,t}^c$ is agent $n$'s consumption growth exposure to the local extrinsic shock $dZ_{n,t}$. Consumption growth is exposed to extrinsic shocks because local equity is exposed and agents cannot share this risk with other locations (e.g., they cannot make cross-location equity market trades). Such risk is *idiosyncratic*, because it necessarily aggregates to zero across locations (i.e., $\sum_{n=1}^{N} x_{n,t}\sigma_{n,t}^c M_n = 0$, because aggregate consumption $Y_t$ is not exposed to extrinsic shocks). As in classical models of exogenous idiosyncratic risks, all agents want to save to self-insure against this idiosyncratic risk, which has the effect of reducing $r_t$ (Bewley, 1986; Huggett, 1993; Aiyagari, 1994).

In our log utility ($\rho = 1$) example from Proposition 1, this idiosyncratic precautionary savings term becomes

$$\sum_{n=1}^{N} x_{n,t}(\sigma_{n,t}^c)^2 = \sum_{n=1}^{N} \frac{1}{x_{n,t}}(\delta\alpha_{n,t}q_{n,t}\sigma_{n,t}^q)^2 = (\delta\psi_t)^2 \sum_{n=1}^{N} \frac{(v_n^*)^2}{x_{n,t}}$$

A rise in the volatility factor $\psi_t$ increases all agents' idiosyncratic risks, which increases the precautionary savings motive.

The poorest agents (i.e., locations with low $x_{n,t}$) have the highest marginal utility and are thus most sensitive to a rise in volatility. In equilibrium, these poor agents will decrease their consumption to pay down existing debt balances as $\psi_t$ rises, while richer agents will consume more by reducing their savings. To see this dynamic, examine the expected consumption growth rate of each location in equilibrium:

$$\mu_{n,t}^c = r_t - \delta + \nu^2 + (\frac{\delta\psi_t v_n^*}{x_{n,t}})^2$$

(This is simply agent $n$'s Euler equation, with extrinsic consumption volatility $\sigma_{n,t}^c$ substituted in.) If $\psi_t$ rises, consumption growth rises in poor locations (those with small $x_{n,t}$) and falls in rich locations (high $x_{n,t}$). As suggested earlier, this happens because poor locations strongly increase their precautionary savings when idiosyncratic risk rises.

### 3.6 The role of incomplete markets

Our model has market incompleteness due to cross-sectionally segmented equity markets. It turns out that this is not necessary; self-fulfilling volatility can emerge even with complete markets, although it can have no *real effects*.

The reader may expect the First Welfare Theorem to hold with complete markets, so how could self-fulfilling volatility emerge? Intuitively, one can understand our function $\Gamma(\cdot)$, which encodes a causal impact of asset prices on endowments, as a pecuniary externality. Such externalities cause deviations from the First Welfare Theorem and allow equilibrium non-uniqueness. Mechanically, if Eq. (19) holds, then the Euler equation for equity valuations is still Eq. (20) in a deterministic equilibrium. This is a stable dynamical system, so it allows some amount of sunspot volatility.

**Proposition 2.** *Suppose financial markets are complete. Let $\rho = 1$, and assume either $N \geq 3$ or $\hat{v} = 0$. Assume growth rates satisfy Eq. (19). Then, an equilibrium exists with self-fulfilling volatility. In fact, the conclusions of Proposition 1 go through without modification.*

*Proof.* See Appendix B.8. □

The critical difference between complete markets and incomplete markets is whether sunspot volatility has real effects. Complete markets feature perfect risk sharing, so the unique SDF $\xi_t$ takes the following form:

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \eta_t dB_t, \quad \text{under complete markets.}$$

The SDF cannot load on the extrinsic shocks $d\tilde{Z}_t$. Furthermore, agents consume according to the rule $e^{-\delta t} c_{n,t}^{-\rho} = \xi_t$, so consumption dynamics $dc_{n,t}$ are conditionally independent from $d\tilde{Z}_t$. Thus, even if asset prices feature non-fundamental volatility, neither risk premia nor consumption dynamics are affected. And the equilibrium interest rate will be given by the "representative-agent" term $r_t = \delta + \rho g_t - \frac{1}{2}\rho(\rho+1)v^2$ in Eq. (25).

On the other hand, with incomplete markets, self-fulfilling volatility in local equity markets induces volatility in consumption, hence non-zero risk premia. To see this, consider the segmented-markets model with log utility ($\rho = 1$). In that model, we show that the location-$n$ SDF follows (see Appendix A)

$$\frac{d\xi_{n,t}}{\xi_{n,t}} = -r_t dt - \eta_t dB_t - \pi_{n,t} dZ_{n,t}, \quad \text{under segmented markets,}$$
$$\text{where} \quad \pi_{n,t} = \delta\Big(\frac{\alpha_{n,t} q_{n,t}}{x_{n,t}}\Big)\sigma^q_{n,t} \quad \text{if } \rho = 1.$$

In segmented markets, the risk price (Sharpe ratio) $\pi_{n,t}$ associated with the extrinsic shock $dZ_{n,t}$ is positive if and only if $\sigma^q_{n,t} > 0$. This risk pricing emerges from consumption risk, since $Y_t \alpha_{n,t} q_{n,t} \sigma^q_{n,t}$ is agent $n$'s total wealth exposure to the extrinsic $dZ_{n,t}$ shock. The presence of a positive risk price for extrinsic shocks is what explains the "risk premium" term augmenting valuation dynamics in Eq. (22). At its source are the self-fulfilling idiosyncratic shocks hitting agents' consumption, which also explains the "idiosyncratic precautionary savings" term arising in $r_t$ in Eq. (25).

While the complete-markets setting is theoretically purer in identifying a set of minimal necessary assumptions that lead to self-fulfilling volatility, we embrace a segmented-markets setting for two reasons. First, segmented markets will be a realistic assumption in our applications below (Section 4). Second, the segmentation-induced consumption dynamics allow us to speak to more puzzles in the literature.

### 3.7 Alternative sources of endogeneity and stability

By now, it should be clear that endogenous growth rates are essential. Having understood that the role of endogenous growth is to induce stable dynamical systems, a natural question is whether alternative sources of endogeneity might work similarly. Internet Appendix C provides three additional examples of endogeneity that also work as "stabilizing forces."

In Internet Appendix C.1, we show that valuation-dependent beliefs can create a stable dynamical system and hence support self-fulfilling volatility. In particular, we suppose that, while true growth rates remain constant, investors become more optimistic about growth when valuations rise. Perhaps agents use asset prices to construct their beliefs about growth to simplify a complex underlying filtering problem, or perhaps rising asset prices just create euphoria amongst investors. Either way, such optimism about growth encourages asset demand which fulfils the initial conjecture of a higher valuation. This specification mirrors our baseline model's growth-valuation link, but only in investors' heads. An interesting outcome is that beliefs are endogenously extrapolative (Barberis et al., 2015).

In Internet Appendix C.2, we show that under-investment, of the type induced by "debt overhang" (e.g., Hennessy, 2004; DeMarzo et al., 2012), creates the needed stability. The main idea is that potential gains from investment are high relative to actual investment, which leaves some surplus on the table. As prices rise and boost investment, debt overhang problems shrink, and some of this surplus is captured by local investors. The extra returns gained this way compensate investors for lower dividend yields and ensure stable price-dividend ratios. An intriguing implication is that under-investment can be a self-fulfilling phenomenon for reasons other than those previously identified (e.g., non-convex technologies or borrowing constraints).

In Internet Appendix C.3, we show that an overlapping generations economy with "creative destruction" (e.g., Aghion and Howitt, 1992; Gârleanu and Panageas, 2020) also produces the required stability. Creative destruction here is represented as new firms entering and displacing incumbents. If the amount of creative destruction is itself a function of asset prices, high asset prices can be self-fulfilled by a reduction in new firm entry, and vice versa. High valuations reduce dividend yields to investors, but living cohorts are compensated with the preservation of their firms, which removes the need for valuations to continue growing and thus creates stability.

Economically, Eq. (19) and the examples in Internet Appendix C share a common property: when prices rise so that dividend yields fall, investors are compensated somehow. This compensation can take the form of higher dividend growth rates, higher perceived growth rates, a drop in under-investment wedges, or less creative destruction. It is likely that many other examples of stabilizing forces also exist. By identifying several, we stress that a wide range of plausible environments all generate a similar type of stability that can support self-fulfilling volatility.

# 4 Applications and extensions

In this section, we discuss three applications of self-fulfilling volatility. The first application considers "locations" to be firms and explores the growth and risk premium consequences of excess idiosyncratic volatility. The second application explores arbitrage profits in a setting where "locations" are distinct and segmented financial markets for trading an identical cash flow stream. The third application interprets "locations" as countries in an international macroeconomy, which features excess volatility of exchange rates and can speak to some puzzles in international finance. In these latter two applications, we will tweak the baseline model to make our interpretation fit in a more natural way. For all results of this section, we assume consumers have log utility ($\rho = 1$).

## 4.1 Firm-specific risks and undiversified insiders

In this section, we interpret each "location" $n$ as a firm, and "representative investor" $n$ as the corporate insiders of that firm (e.g., CEOs). In fact, firm insiders are often not fully diversified (May, 1995; Guay, 1999; Himmelberg et al., 2002; Panousi and Papanikolaou, 2012) and their individual preferences and other characteristics seem to matter in firms' decision processes (Bertrand and Schoar, 2003; Graham et al., 2013). Such concentrated risk exposure can arise as an optimal pay-for-performance compensation contract in the presence of moral hazard or signalling/selection issues (Holmström, 1979; Leland and Pyle, 1977; Rock, 1986). Our model partly captures this phenomenon, because the location-$n$ equity market is segmented from other locations, and its risk is borne by the location-$n$ investors. We say "partly" because our investors have access to a futures market that allows them to share risks from the location-specific fundamental shocks $d\hat{B}_t$. If we wanted to better capture a setting of corporate insiders, we could also eliminate this particular futures market, in which case the insiders would effectively be holding a portfolio of their firm's equity along with outside borrowing/lending (position in riskless bonds) and trading in the aggregate stock market index (futures on $dB_t$).[7]

With the model applied to firms, many microfoundations of a connection between valuations and fundamentals seem plausible. An endogeneity of cash flow growth rates, captured by $\Gamma(q)$ in the baseline model, can be thought of here as "feedback effects" between stock prices and investment (Bond et al., 2012). Alternatively, as discussed in Internet Appendix C.2, one could interpret these firms as having debt outstanding, in which case debt-overhang problems lead to a connection between valuations and investment efficiency. Either of these interpretations seem appropriate for firms, and both foster self-fulfilling volatility.

Self-fulfilling volatility is necessarily idiosyncratic, in that it aggregates to zero, as emphasized by Lemma 4. Yet this idiosyncratic volatility features a *common component*: firm $n$ self-fulfilling return volatility is $\sigma^q_{n,t} = \psi_t v^*_n / \alpha_{n,t} q_{n,t}$, which scales with the com-

[7] In such an extension, the key modification would be that fundamental idiosyncratic risk demands a risk premium from undiversified insiders. Mathematically, the absence of a futures market for $d\hat{B}_t$ implies $\hat{\vartheta}_{n,t} = 0$ for each $n$, which implies insider consumption growth has the following exposure to $d\hat{B}_t$:

$$\frac{1}{dt}\mathrm{Cov}_t[\frac{dc_{n,t}}{c_{n,t}}, d\hat{B}_t] = \frac{\delta \alpha_{n,t} q_{n,t}}{x_{n,t}}(\hat{v}_{n,t} + \hat{\varsigma}_{n,t}).$$

(See Eq. (A.12) in Appendix A with $\hat{\vartheta}_{n,t} = 0$ substituted.) Consequently, the expected capital gain in Eq. (A.2) must be augmented by the risk premium from this exposure, namely $\frac{\delta \alpha_{n,t} q_{n,t}}{x_{n,t}} \|\hat{v}_{n,t} + \hat{\varsigma}_{n,t}\|^2$. This modification would substantially complicate the type of equilibrium construction done in Proposition 1, because now the dynamics of $q_{n,t}$ become coupled with those of $\alpha_{n,t}$ and $x_{n,t}$, even when valuation volatility vanishes. However, the spirit of the analysis does not change.

mon factor $\psi_t$. In the data, Campbell et al. (2001) and Herskovic et al. (2016) document a significant and highly time-varying common component in idiosyncratic return volatility. The existence of this common component is one of the most important implications of our model, since it provides a plausible microfoundation to researchers that have modeled exogenously time-varying idiosyncratic volatility and its macroeconomic effects (Di Tella, 2017, 2020; Di Tella and Hall, 2022).

Not only should idiosyncratic stock returns contain a common factor, fundamentals should too. Indeed, firm-level growth rates are influenced by stock valuations, through the function $\Gamma(q)$. Firms that are doing particularly well in the stock market should also have particularly high investment and growth rates. Firms doing poorly should be "underinvesting." This spread between firm-level growth rates is also magnified by the common volatility factor $\psi_t$.

The firm dynamics literature (Hopenhayn, 1992; Sutton, 1997; Luttmer, 2007; Gabaix, 2009) has emphasized random log-normal growth (plus a "friction") as a possible reason for the fat-tailed firm size distribution. One quantitative difficulty has been explaining the thickness of the tail with realistic levels of firm-specific volatility. Our framework can alleviate this issue, since larger firms will tend to grow faster. In general, a positive correlation between size and growth rates will magnify the size dispersion in any real variable such as sales.

Although it is idiosyncratic, self-fulfilling volatility commands a risk premium, because insiders hold concentrated, undiversified exposures to their own stocks. The idiosyncratic risk premium for firm-$n$ equity is given by

$$\sigma^q_{n,t}\pi_{n,t} = \frac{\delta(\psi_t v_n^*)^2}{x_{n,t}\alpha_{n,t}q_{n,t}}.$$

When self-fulfilling volatility spikes ($\psi_t$), measured risk premia also rise. In the data, Herskovic et al. (2016) find that the common component in idiosyncratic volatility is priced, consistent with this implication.

## 4.2 Arbitrage and segmentation

In this section, we explore how self-fulfilling volatility relates to the existence of arbitrage profits. We consider a setting with two assets having identical cash flows but potentially different prices due to market segmentation. To entertain this possibility, it becomes necessary to modify our model somewhat. Rather than a growth-valuation link, we introduce a *belief-valuation link*: high valuations in market $n$ lead to optimism about the cash flow growth for asset $n$. With this modification, asset valuations can exhibit self-fulfilling volatility, even if their cash flows are identical in reality.

We briefly outline the extended model, with full details in Internet Appendix C.1. The cash flows of assets $n \in \{1, 2\}$ are identical geometric Brownian motions

$$\frac{dy_{n,t}}{y_{n,t}} = gdt + \nu dB_t.$$

There are no fundamental idiosyncratic shocks ($\hat{\nu} = 0$) and no endogeneity in growth rates ($\Gamma \equiv 0$). To allow non-fundamental fluctuations in this setting, we model subjective beliefs as follows. Under their subjective probability $\widetilde{\mathbb{P}}^n$, agents in location $n$ believe that $d\tilde{B}_{n,t} := dB_t - \gamma_{n,t}dt$ is a Brownian motion, where

$$\gamma_{n,t} = \frac{\lambda}{\nu}(q_{n,t} - \delta^{-1}). \tag{26}$$

An implication of this assumption is that agent $n$ holds the following subjective belief $\tilde{g}_{n,t} := \frac{1}{dt}\widetilde{\mathbb{E}}_t^n[\frac{dy_{n,t}}{y_{n,t}}]$ about his local endowment growth rate:

$$\tilde{g}_{n,t} = g + \lambda(q_{n,t} - \delta^{-1}). \tag{27}$$

Eq. (27) mirrors Eq. (19), but for perceived growth rather than true growth. When valuations rise, investors demand their local asset more because they wrongly expect its cash flows to grow faster, so valuations can slowly fall back to "steady state" and still satisfy investors' required returns. For this intuition to hold, we must also eliminate the integrated futures markets, i.e., there is no cross-location trading of the $dB_t$ shock. With perfect aggregate risk-sharing, all investors would agree on the aggregate risk price, severing the feedback between beliefs and valuation dynamics.

In Internet Appendix C.1, we show that this model can exhibit self-fulfilling volatility. To facilitate the analysis and prove this formally, the appendix studies a limiting case where one of the two locations is vanishingly small relative to the other, akin to a "small open economy." The large asset ($n = 2$) has a constant price-dividend ratio $q_{2,t} = \delta^{-1}$, while valuation ratio for the small asset ($n = 1$) can fluctuate in a self-fulfilling way, with

$$q_{1,t}\sigma_{1,t}^q = \psi_t$$

for some positive process $\psi_t$. As in Proposition 1, the volatility process $\psi_t$ must vanish far away from steady state, so that the endogenous beliefs push valuations back toward steady state.

Here, the existence of self-fulfilling volatility implies an *arbitrage*. Two assets with identical cash flows can have different valuations, constituting a violation of the law of one price. The empirical literature has documented several such arbitrage trades. Examples include spinoffs (Lamont and Thaler, 2003); "on-the-run/off-the-run" bonds (Krishnamurthy, 2002); put-call parity (Ofek et al., 2004); convertible bonds (Mitchell, Pedersen and Pulvino, 2007); covered interest parity (Du, Tepper and Verdelhan, 2018; Du, Hébert and Huber, 2019); Treasury spot and future-implied repo rates (Fleckenstein and Longstaff, 2018); and cryptocurrencies (Makarov and Schoar, 2020).

Typically, such law of one price violations require two imperfections: (i) random non-fundamental asset demand and (ii) limits-to-arbitrage frictions. Random asset demand potentially pushes apart valuations of identical assets apart, while limits-to-arbitrage prevents these valuations from quickly converging. For instance, in a seminal paper, Gromb and Vayanos (2002) model random demand from local hedgers, along with arbitrageurs constrained by margin requirements. Common alternatives are to replace "local hedgers" by "noise traders" (De Long et al., 1990a,b, 1991; Kyle and Xiong, 2001; Vayanos and Vila, 2021) and replace margin requirements by myopic performance-based clients (Shleifer and Vishny, 1997) or search frictions (Vayanos and Weill, 2008; Duffie and Strulovici, 2012). Relative to this extant literature, our contribution is to show how "randomness" in asset demand can be a self-fulfilling outcome of coordination, while we remain relatively silent on any particular microfoundation for the segmentation between asset markets (Dávila et al., 2021 take a similar approach in remaining agnostic about the specific constraints limiting arbitrage).

How profitable are the arbitrage trades? Suppose $q_{1,t} < \delta^{-1} = q_{2,t}$. Consider a zero-cost long-short strategy that buys asset 1 and shorts asset 2. This strategy's excess returns are $\frac{dq_{1,t}}{q_{1,t}} - \frac{dq_{2,t}}{q_{2,t}} + [\frac{1}{q_{1,t}} - \frac{1}{q_{2,t}}]dt$, which in equilibrium equals (see the expressions in Internet Appendix C.1)

$$\Big( \underbrace{\lambda(\delta^{-1} - q_{1,t})}_{\text{pessimism}} + \underbrace{\sigma^q_{1,t}\text{Cov}_t[\frac{dc_{1,t}}{c_{1,t}}, dZ_t]}_{\substack{\text{self-fulfilling} \\ \text{risk premium}}} + \underbrace{\nu\text{Cov}_t[\frac{dc_{1,t}}{c_{1,t}} - \frac{dY_t}{Y_t}, dB_t]}_{\substack{\text{imperfect aggregate} \\ \text{risk-sharing}}} \Big) dt + \sigma^q_{1,t} dZ_t. \tag{28}$$

On average, the trade makes money, due primarily to the term $\lambda(\delta^{-1} - q_{1,t}) > 0$ labeled "pessimism." Location-1 investors become pessimistic when their local asset is cheap, so their asset must appreciate in value at rate $\lambda$ to satisfy their required returns. This trade is not riskless, however: future self-fulfilling shocks ($\sigma^q_{1,t} dZ_t$) can push valuations further apart. Because most empirical arbitrage trades are hold-to-maturity strategies, they are also risky in the interim, despite having large expected excess returns.[8] The other terms in (28) are less critical to this particular application and reflect risk premia owed to the local investors.[9]

A reasonable view might be that the profitability of arbitrage trades cannot be too large. For example, "good-deal" Sharpe ratio bounds (Cochrane and Saa-Requejo, 2000) suggest there should be some maximum Sharpe ratio $\Pi_t$ that arbitrageurs enforce, in which case our equilibrium will require

$$\left|\lambda(\delta^{-1} - q_{1,t}) + \sigma^q_{1,t}\text{Cov}_t[\frac{dc_{1,t}}{c_{1,t}}, dZ_t] + \nu\text{Cov}_t[\frac{dc_{1,t}}{c_{1,t}} - \frac{dY_t}{Y_t}, dB_t]\right| \leq \Pi_t\sigma^q_{1,t}$$

While we do not take a stand on what drives the level and dynamics of $\Pi_t$, candidates in the literature include value-at-risk, margin, and regulatory constraints, which imply that $\Pi_t$ should be positively related to asset volatility and capital requirements and inversely related to arbitrageur wealth (Adrian and Shin, 2010; Gârleanu and Pedersen, 2011; Boyarchenko et al., 2018).

## 4.3 International macro and exchange rates

Our final application interprets "locations" as countries, each of which has its own fundamentals. When thinking about international finance, equity market segmentation is a fact of life that induces market incompleteness and can potentially speak to some puzzling observations (Gabaix and Maggiori, 2015; Lustig and Verdelhan, 2019; Itskhoki and Mukhin, 2021). We will discuss how our model, simply through non-fundamental fluctuations in asset prices, can help address excess exchange rate volatility (e.g., the PPP puzzle), international risk-sharing puzzles (e.g., Backus-Smith puzzle), and carry trade returns (e.g., UIP puzzle). Because our non-fundamental volatility has a factor structure, it also connects to an international finance literature that has discovered a latent factor governing the lion's share of exchange rate and global financial market movements.

To tailor our model to the international setting, we introduce a non-tradable endowment $\hat{y}_{n,t}$. For simplicity and theoretical clarity on what is new with our framework, we assume $\hat{y}_{n,t}$ follows the same time-series growth process as the tradable $y_{n,t}$ in Eq. (1); in particular, let $\hat{y}_{n,t} = \kappa y_{n,t}$ for all $n$. The representative agent of country $n$ consumes

[8] One way of modeling these hold-to-maturity arbitrage trades as risky yet high-returning "convergence trades" are via the Brownian bridge process (Liu and Longstaff, 2004).

[9] In particular, the asset-1 risk $\sigma^q_{1,t}$ requires a risk premium $\pi_{1,t}\sigma^q_{1,t}$ that is earned in this case where the arbitrageur goes long asset 1, whereas this risk premium would be paid in the reverse strategy. The third term—which is the excess risk exposure of location-1 investors to fundamental shocks and reflects imperfect aggregate risk-sharing—is zero on average.

$\hat{c}_{n,t}$ of the non-tradable, which in equilibrium is $\hat{c}_{n,t} = \hat{y}_{n,t}$. The tradable market still clears globally via $\sum_{n=1}^{N} c_{n,t} = \sum_{n=1}^{N} y_{n,t}$. Agents have preferences over a Cobb-Douglas aggregate of tradables and their local non-tradable, i.e.,

$$\mathbb{E}_0\Big[\int_0^\infty e^{-\delta t}\Big(\phi\log(c_{n,t}) + (1-\phi)\log(\hat{c}_{n,t})\Big)dt\Big]. \tag{29}$$

We set the tradable good as the numéraire, so let $p_{n,t}$ be the relative price of the country $n$ non-tradable. We let $q_{n,t}$ still be the local valuation ratio, so that the total value of the local endowment is $q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})$. Finally, we assume a growth-valuation link $\Gamma(q)$ according to the linear functional form (19), so that the country $n$ output growth rate is $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$.

Besides incomplete financial markets, this setting is identical to Backus and Smith (1993) and many other studies. The solution is as follows. In this model, the consumption basket and price index of country $n$ are given by

$$\begin{aligned} C_{n,t} &:= c_{n,t}^{\phi}\hat{c}_{n,t}^{1-\phi} \\ P_{n,t} &:= \frac{c_{n,t} + p_{n,t}\hat{c}_{n,t}}{C_{n,t}}. \end{aligned}$$

The total expenditure of country $n$ is thus $P_{n,t}C_{n,t}$. Because of log utility, agents optimally spend $\delta$ fraction of their wealth, so

$$P_{n,t}C_{n,t} = \delta w_{n,t}. \tag{30}$$

Cobb-Douglas period utility implies the optimal expenditure shares of tradables and non-tradables are $\phi$ and $1-\phi$, respectively:

$$c_{n,t} = \phi P_{n,t}C_{n,t} \quad \text{and} \quad p_{n,t}\hat{c}_{n,t} = (1-\phi)P_{n,t}C_{n,t}. \tag{31}$$

Using Eqs. (30)-(31) and non-tradable market clearing $\hat{c}_{n,t} = \hat{y}_{n,t}$, the price index can be written

$$P_{n,t} = \phi^{-1}\Big(\frac{c_{n,t}}{\hat{y}_{n,t}}\Big)^{1-\phi} = \phi^{-\phi}\Big(\frac{\delta w_{n,t}}{\hat{y}_{n,t}}\Big)^{1-\phi}.$$

The *real exchange rate* $\mathcal{E}_t^{i,j}$ between countries $i$ and $j$, defined as the ratio of their price indexes, is

$$\mathcal{E}_t^{i,j} := \frac{P_{j,t}}{P_{i,t}} = \Big(\frac{x_{j,t}/x_{i,t}}{\hat{y}_{j,t}/\hat{y}_{i,t}}\Big)^{1-\phi}, \tag{32}$$

where $x_{i,t}$ is the tradable consumption share of country $i$ (because of log utility, $x_{i,t}$ is equivalently the wealth share of country $i$).

The remainder of equilibrium is very similar to the baseline model without non-tradables. Most importantly, there exist non-fundamental equilibria in which the valuation ratios $(q_{n,t})_{n=1}^N$ are hit by sunspot fluctuations that necessarily redistribute wealth across countries. Such wealth redistribution means that $(x_{n,t})_{n=1}^N$ are also subject to sunspot fluctuations. The full details of equilibrium derivation with non-tradables are in Internet Appendix D.

The sunspot equilibria of this model are helpful in resolving a few exchange rate puzzles. First, real exchange rates in Eq. (32) inherit additional sources of volatility from the wealth shares $(x_{n,t})_{n=1}^N$. Indeed, the dynamics of $x_{n,t}$ are given by

$$\frac{dx_{n,t}}{x_{n,t}} = \Big(\pi_{n,t}^2 - \sum_{i=1}^{N} x_{i,t}\pi_{i,t}^2\Big)dt + \pi_{n,t}dZ_{n,t}. \tag{33}$$

Because wealth shares are driven by extrinsic shocks, our model features higher volatility of the real exchange rate over and above macroeconomic aggregates. This provides a partial resolution to the classic volatility puzzles of Meese and Rogoff (1983) and Mussa (1986).[10] In terms of the direction, our model predicts a positive link between capital flows and exchange rates, as in Gabaix and Maggiori (2015): a positive extrinsic shock $dZ_{n,t} > 0$ induces a capital flow into country $n$ from the rest of the world (so that $c_{n,t}$ can rise above $y_{n,t}$), which causes an appreciation of country $n$'s real exchange rate.

Second, sunspot volatility helps break a tight positive link between exchange rates and relative aggregate consumptions across countries, providing some resolution to the Backus and Smith (1993) puzzle (see also Kollmann, 1995 and Corsetti et al., 2008).[11] Different to complete-market models, sunspot shocks in our incomplete-markets model actually induce a negative comovement between exchange rates and relative consump-

[10] Meese and Rogoff (1983) show that the nominal exchange rate is significantly more volatile than macroeconomic aggregates like consumption and output, while Mussa (1986) shows that the real and nominal exchange rate behaviors are tightly linked. See also the survey in Rogoff (1996) on the Purchasing Power Parity (PPP) puzzle.

[11] To compare to these papers, use Eq. (30) to write the exchange rate in terms of the aggregate consumption baskets:

$$\mathcal{E}_t^{i,j} = \frac{C_{i,t}}{C_{j,t}}\frac{w_{j,t}}{w_{i,t}}.$$

Critically, the wealth ratio $w_{j,t}/w_{i,t} = x_{j,t}/x_{i,t}$ is not constant in our incomplete-markets model. By contrast, in any complete-markets, symmetric preference model, the wealth distribution is constant.

tions. To see this, notice that the relative aggregate consumptions can be written

$$\frac{C_{i,t}}{C_{j,t}} = \Big(\frac{c_{i,t}}{c_{j,t}}\Big)^{\phi} \Big(\frac{\hat{c}_{i,t}}{\hat{c}_{j,t}}\Big)^{1-\phi} = \Big(\frac{x_{i,t}}{x_{j,t}}\Big)^{\phi} \Big(\frac{\hat{y}_{i,t}}{\hat{y}_{j,t}}\Big)^{1-\phi}.$$

The critical observation is that $C_{i,t}/C_{j,t}$ is increasing in the wealth ratio $x_{i,t}/x_{j,t}$, whereas the exchange rate is decreasing in this wealth ratio—see Eq. (32). Therefore, the presence of extrinsic shocks that move the wealth distribution can substantially reduce the correlation between $\mathcal{E}_t^{i,j}$ and $C_{i,t}/C_{j,t}$. (Note that without the wealth distribution dynamics, our model would have $\mathcal{E}_t^{i,j} = C_{i,t}/C_{j,t}$, a particularly stark representation of the Backus and Smith (1993) puzzle.)

The puzzles so far are about bilateral exchange rates and their "disconnect" from macroeconomic fundamentals. But this disconnect also has a factor structure. In our equilibria, the volatilities of all bilateral exchange rates all rise and fall together, since there is a common factor $\psi_t$ in sunspot volatility, i.e.,

$$\pi_{n,t} = \frac{\delta \psi_t}{x_{n,t}} v_n^*. \tag{34}$$

The common volatility factor $\psi_t$ could be related to the empirical factor structure in exchange rates, documented in Lustig et al. (2011) and theorized in Gourio et al. (2013) to be linked to equity volatility. Because $\psi_t$ governs the volatilities of all countries' equity markets, this factor could also be related to the global financial cycle documented in Rey (2015) and linked to the VIX.

Finally, our self-fulfilling fluctuations help explain carry trade returns and uncovered interest parity (UIP) deviations (Fama, 1984; McCallum, 1994; Engel, 1996). We compute the price of a pure discount bond that pays off one unit of the country-$n$ consumption basket:

$$b_{n,t\to T} := \mathbb{E}_t\Big[\frac{\xi_{n,T}}{\xi_{n,t}} \frac{P_{n,T}}{P_{n,t}}\Big]$$

Note that, due to the normalization by $P_{n,t}$, the price $b_{n,t\to T}$ is denominated in units of the country-$n$ consumption basket. One can then show that the yield-to-maturity of this bond, $\mathrm{YTM}_{n,t\to T} := -\frac{1}{T-t}\log(b_{n,t\to T})$, is given by[12]

$$\mathrm{YTM}_{n,t\to T} = \delta - \frac{1}{T-t} \log \mathbb{E}_t\Big[\Big(\frac{x_{n,t}}{x_{n,T}}\Big)^{\phi} \Big(\frac{\hat{y}_{n,t}}{\hat{y}_{n,T}}\Big)^{1-\phi} \Big(\frac{Y_t}{Y_T}\Big)^{\phi}\Big], \tag{35}$$

[12]To derive this equation, substitute the price index $P_{n,t} = \phi^{-1}(c_{n,t}/\hat{y}_{n,t})^{1-\phi}$ and use the optimal consumption rule $\xi_{n,t} = e^{-\delta t}\omega c_{n,t}^{-1}$.

The key observation is that countries with high expected wealth share growth—i.e., countries with high $\pi_{n,t}$, or equivalently low $x_{n,t}$ by Eq. (34)—will have high bond yields. It turns out that these same countries have high UIP deviations. Indeed, the expected carry return going long the country-$j$ bond and short the country-$i$ bond is

$$R^{i,j}_{t\to T} := \underbrace{\mathrm{YTM}_{j,t\to T} - \mathrm{YTM}_{i,t\to T}}_{>0 \text{ if } x_{j,t} \ll x_{i,t}} + \frac{1}{T-t} \underbrace{\mathbb{E}_t[\log \mathcal{E}^{i,j}_T - \log \mathcal{E}^{i,j}_t]}_{>0 \text{ if } x_{j,t} \ll x_{i,t}}. \tag{36}$$

Since $\mathcal{E}^{i,j}_t$ is increasing in the wealth ratio $x_{j,t}/x_{i,t}$, countries with high expected wealth growth will experience an expected appreciation of their exchange rate. Such an expected appreciation further exacerbates the expected carry return $R^{i,j}_{t\to T}$ beyond what is predicted by the yield advantage of country $j$.[13]

# 5 Conclusion

This paper provides a theory of self-fulfilling fluctuations that are redistributive in nature. Theoretically, the existence of such self-fulfilling volatility relies on multiple segmented markets and an endogenous force that connects asset valuations to some aspect of the real economy—our baseline model studies a valuation-growth link, but alternatives studied in the Internet Appendix include beliefs about growth rates (as in "price extrapolation" models), underinvestment wedges (as in "debt overhang" models), and entry/exit patterns (as in "creative destruction" models). We refer to these sources of endogeneity as "stabilizing forces" because their role is keep valuations stationary in sunspot equilibria; one can imagine several other examples of stabilizing forces might exist. Our framework helps explain the factor structure in firm-specific volatility, the existence arbitrage trades and their profitability, and various dimensions of exchange rate disconnect such as the PPP puzzle, the Backus-Smith puzzle, and the UIP puzzle.

[13] In this discussion, we have ignored the effects of $\hat{y}_{i,t}$ and $\hat{y}_{j,t}$, because their role in bond yields is offset by their role in exchange rates. To see this transparently, suppose as an approximation we take

$$\log \mathbb{E}_t \Big[ \Big(\frac{x_{n,t}}{x_{n,T}}\Big)^{\phi} \Big(\frac{\hat{y}_{n,t}}{\hat{y}_{n,T}}\Big)^{1-\phi} \Big(\frac{Y_t}{Y_T}\Big)^{\phi} \Big] \approx \mathbb{E}_t \log \Big[ \Big(\frac{x_{n,t}}{x_{n,T}}\Big)^{\phi} \Big(\frac{\hat{y}_{n,t}}{\hat{y}_{n,T}}\Big)^{1-\phi} \Big(\frac{Y_t}{Y_T}\Big)^{\phi} \Big].$$

Then, under this approximation, we have

$$R^{i,j}_{t\to T} \approx \frac{1}{T-t} \mathbb{E}_t \Big[ \log\big(\frac{x_{j,T}}{x_{j,t}}\big) - \log\big(\frac{x_{i,T}}{x_{i,t}}\big) \Big] = \frac{1}{T-t} \mathbb{E}_t \int_t^T \frac{\pi^2_{j,s} - \pi^2_{i,s}}{2} ds.$$

Thus, approximately the entire UIP deviation emerges due to wealth distribution dynamics.

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# Appendix

## A Derivation of Equilibrium

In this appendix, we derive the complete set of equilibrium conditions that will be used throughout the entire analysis.

**Step 1: State prices.** Each location has its own state-price density $\xi_{n,t}$, which follows

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \eta_t dB_t + \hat{\eta}_t \cdot d\hat{B}_t + \pi_{n,t} dZ_{n,t}\Big]. \tag{A.1}$$

The market prices of risk $(\eta, \hat{\eta})$ associated to $(B, \hat{B})$ are location-invariant, because markets for trading futures on these shocks are perfectly integrated. The contributions of extrinsic shocks is limited to $\pi_{n,t} dZ_{n,t} = \pi_{n,t} M_n d\tilde{Z}_t$, i.e., the extrinsic shock vector $\tilde{Z}$ only matters through the weighted-sum $M_n d\tilde{Z}_t$, because this is exactly the shock that affects the local asset price $q_{n,t}$. In principle, this is a restriction that we may not need to make, but it will turn out to be sufficient for our purposes—in other words, a large variety of equilibria will exist consistent with (A.1).

In terms of these state prices, we have the no-arbitrage pricing relation for location-$n$ equity:

$$\mu^q_{n,t} + g_{n,t} + \frac{1}{q_{n,t}} + \nu \varsigma^q_{n,t} + \hat{\nu}_{n,t} \cdot \hat{\varsigma}^q_{n,t} - r_t = (\nu + \varsigma^q_{n,t})\eta_t + (\hat{\nu}_{n,t} + \hat{\varsigma}^q_{n,t}) \cdot \hat{\eta}_t + \sigma^q_{n,t} \pi_{n,t}, \tag{A.2}$$

where with some abuse of notation we have defined the idiosyncratic risk exposure vector for $y_{n,t}$,

$$\hat{\nu}_{n,t} := \hat{\nu}\left[e_n - \begin{pmatrix} \alpha_{1,t} \\ \vdots \\ \alpha_{N,t} \end{pmatrix}\right] = \frac{1}{dt}\mathrm{Cov}_t[\frac{dy_{n,t}}{y_{n,t}}, d\hat{B}_t], \tag{A.3}$$

where $e_n$ is the elementary vector having a one in position $n$ and zeros elsewhere, and recall that $\alpha_{n,t} := y_{n,t}/Y_t$ are the endowment shares. Eq. (A.2) suffices to ensure no arbitrage in the equity market, so long as $q_{n,t} > 0$, which must hold in any equilibrium by free-disposal. The endowment share evolution is derived by applying Itô's formula to the definition of $\alpha_{n,t}$, namely

$$\frac{d\alpha_{n,t}}{\alpha_{n,t}} = (g_{n,t} - g_t)dt + \hat{\nu}_{n,t} \cdot d\hat{B}_t. \tag{A.4}$$

**Step 2: Optimality.** Integrating the dynamic budget constraint (3), using state-price dynamics (A.1), the pricing Eq. (A.2), and the individual transversality condition

$$\lim_{T\to\infty} \mathbb{E}_t[\xi_{n,T} w_{n,T}] = 0, \tag{A.5}$$

we obtain the standard static budget constraint

$$\mathbb{E}_t\Big[\int_t^\infty \frac{\xi_{n,s}}{\xi_{n,t}} c_{n,s} ds\Big] = w_{n,t}. \tag{A.6}$$

Note in passing that (A.6) implies $w_{n,t} > 0$, so the solvency constraint holds automatically. Agents' optimization problem is thus simply to maximize (5) subject to (A.6). The first-order condition of this optimization problem is

$$e^{-\delta t} c^{-\rho}_{n,t} = \xi_{n,t}. \tag{A.7}$$

Apply Itô's formula to Eq. (A.7) to obtain the following optimal consumption dynamics

$$\frac{dc_{n,t}}{c_{n,t}} = \frac{1}{\rho}\Big[r_t - \delta + \frac{\rho+1}{2\rho}\Big(\eta_t^2 + \|\hat{\eta}_t\|^2 + \pi_{n,t}^2\Big)\Big]dt + \frac{1}{\rho}\Big[\eta_t dB_t + \hat{\eta}_t \cdot d\hat{B}_t + \pi_{n,t} dZ_{n,t}\Big]. \tag{A.8}$$

To solve for the initial consumption $c_{n,t}$, given initial wealth $w_{n,t}$ and the dynamics of state prices and beliefs, substitute (A.7) back into (A.6) to get an equation for the wealth-consumption ratio

$$\omega_{n,t} := \frac{w_{n,t}}{c_{n,t}} = \mathbb{E}_t\Big[\int_t^\infty e^{-\delta(s-t)}\Big(\frac{c_{n,s}}{c_{n,t}}\Big)^{1-\rho} ds\Big]. \tag{A.9}$$

In general, Eq. (A.9) is useful because the dynamics of $c_{n,t}$ are given by Eq. (A.8) in terms of the state price density, so given all asset prices and initial wealth $w_{n,t}$, Eq. (A.9) allows us to compute $c_{n,t}$. (In particular, this will be useful when we study the log utility case with $\rho = 1$, since then Eq. (A.9) collapses to $w_{n,t}/c_{n,t} = \delta^{-1}$.) To instead represent (A.9) as a dynamic evolution equation, suppose

$$d\omega_{n,t} = \omega_{n,t}[\mu_{n,t}^\omega dt + \varsigma_{n,t}^\omega dB_t + \hat{\varsigma}_{n,t}^q \cdot d\hat{B}_t + \sigma_{n,t}^q dZ_{n,t}]$$

and then apply Itô's formula to $\xi_{n,t}\omega_{n,t}c_{n,t} = \mathbb{E}_t[\int_0^\infty \xi_{n,s}c_{n,s}ds] - \int_0^t \xi_{n,s}c_{n,s}ds$, and match drifts to obtain

$$\mu_{n,t}^\omega = \rho^{-1}\delta - \frac{1}{\omega_{n,t}} + \frac{\rho-1}{2\rho^2}\Big[\eta_t^2 + \|\hat{\eta}_t\|^2 + \pi_{n,t}^2\Big] + (1-\rho^{-1})\Big[r_t + \eta_t\varsigma_{n,t}^\omega + \hat{\eta}_t \cdot \hat{\varsigma}_{n,t}^\omega + \pi_{n,t}\sigma_{n,t}^\omega\Big]. \tag{A.10}$$

At the same time, since $\omega_{n,t} = w_{n,t}/c_{n,t}$, the wealth-consumption ratio diffusion coefficients are

$$\varsigma_{n,t}^\omega = \frac{\vartheta_{n,t}}{w_{n,t}} + \frac{\theta_{n,t}}{w_{n,t}}(\nu + \varsigma_{n,t}^q) - \rho^{-1}\eta_t \tag{A.11}$$

$$\hat{\varsigma}_{n,t}^\omega = \frac{\hat{\vartheta}_{n,t}}{w_{n,t}} + \frac{\theta_{n,t}}{w_{n,t}}(\hat{\nu}_{n,t} + \hat{\varsigma}_{n,t}^q) - \rho^{-1}\hat{\eta}_t \tag{A.12}$$

$$\sigma_{n,t}^\omega = \frac{\theta_{n,t}}{w_{n,t}}\sigma_{n,t}^q - \rho^{-1}\pi_{n,t} \tag{A.13}$$

which identifies optimal portfolio choices $(\vartheta_n, \hat{\vartheta}_n, \theta_n)$, given the wealth-consumption volatilities, asset price volatilities, and state price dynamics. Eqs. (A.11)-(A.13) simplify with log utility, since as mentioned earlier the wealth-consumption ratio is constant, $\omega_{n,t} = \delta^{-1}$. For instance, with $\rho = 1$, Eq. (A.13) states that $\theta_{n,t}\sigma_{n,t}^q = w_{n,t}\pi_{n,t}$, so that $\sigma_{n,t}^q > 0$ if and only if $\pi_{n,t} > 0$.

**Step 3: Aggregation.** Recall the consumption shares $x_{n,t} := c_{n,t}/Y_t$. Using (A.8), apply Itô's formula to the goods market clearing condition $\sum_{n=1}^N c_{n,t} = Y_t$, and match drift and diffusion coefficients to obtain an equation for the riskless rate

$$r_t = \delta + \rho g_t - \frac{1}{2}\rho(\rho+1)\nu^2 - \frac{\rho+1}{2\rho}\sum_{n=1}^N x_{n,t}\pi_{n,t}^2 \tag{A.14}$$

expressions for the fundamental risk prices

$$\eta_t = \rho\nu \tag{A.15}$$

$$\hat{\eta}_t = 0 \tag{A.16}$$

and finally an equation linking the extrinsic risk prices (recall: $\pi_{n,t} dZ_{n,t} = \pi_{n,t} M_n d\tilde{Z}_t$ by Eq. (9))

$$0 = \sum_{n=1}^{N} x_{n,t} \pi_{n,t} M_n. \tag{A.17}$$

These expressions are all derived conditional on the consumption shares $(x_{n,t})_{n=1}^{N}$. Consumption share dynamics are obtained by applying Itô's formula to the definition of $x_{n,t}$, with the result being (after substituting several results above)

$$\frac{dx_{n,t}}{x_{n,t}} = \frac{\rho+1}{2\rho^2} \left( \pi_{n,t}^2 - \sum_{i=1}^{N} x_{i,t} \pi_{i,t}^2 \right) dt + \frac{\pi_{n,t}}{\rho} dZ_{n,t}. \tag{A.18}$$

Next, the combination of bond and equity market clearing imply the aggregate wealth constraint

$$\sum_{n=1}^{N} w_{n,t} = \sum_{n=1}^{N} q_{n,t} y_{n,t}. \tag{A.19}$$

Finally, apply equity and futures market clearing conditions to Eqs. (A.11)-(A.13), along with Eq. (A.19) and the expressions for the various risk prices, to obtain

$$\sum_{n=1}^{N} \alpha_{n,t} q_{n,t} \varsigma_{n,t}^{q} = \sum_{n=1}^{N} x_{n,t} \omega_{n,t} \varsigma_{n,t}^{\omega} \tag{A.20}$$

$$\sum_{n=1}^{N} \alpha_{n,t} q_{n,t} (\hat{v}_{n,t} + \hat{\varsigma}_{n,t}^{q}) = \sum_{n=1}^{N} x_{n,t} \omega_{n,t} \hat{\varsigma}_{n,t}^{\omega} \tag{A.21}$$

$$\alpha_{n,t} q_{n,t} \sigma_{n,t}^{q} = x_{n,t} \omega_{n,t} \left[ \rho^{-1} \pi_{n,t} + \sigma_{n,t}^{\omega} \right] \tag{A.22}$$

# B Proofs

## B.1 Existence and Uniqueness Theorem for BSDEs

In the results of this section, let $B$ be a $d$-dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathscr{F}_t$ is the completion of the sigma-algebra generated by $B$. Given a constant $\lambda$, let $\mathscr{L}^{2,\lambda}(0,\infty;\mathbb{R}^n)$ be the Hilbert space of all $\mathbb{R}^n$-valued $\mathscr{F}_t$-adapted processes $v$ such that

$$\mathbb{E}\int_0^\infty e^{\lambda t}|v_t|^2 dt < \infty.$$

In all expressions, $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the usual Euclidean inner product and norms, respectively.

Consider the following backward stochastic differential equation (BSDE):

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_s, \tag{B.1}$$

where the function $f$ is a progressively-measurable mapping,

$$f : \Omega \times [0,\infty) \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$$

Suppose there exist two constants $\alpha$ and $K$ such that $f$ satisfies the following for all $(t, y, z) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}^d$:

(C.i) $(y - y')(f(t, y, z) - f(t, y', z)) \leq -\alpha(y - y')^2$

(C.ii) $|f(t, y, z) - f(t, y', z')| \leq K(|y - y'| + |z - z'|)$

(C.iii) $|f(t, 0, 0)| \leq K$

A solution to the BSDE is a pair $(Y_t, Z_t)_{t\geq 0}$ of progressively-measurable processes such that (B.1) holds on every interval $[t, T]$. The following theorem is a special case of Lemma 3.1 in Briand and Hu (1998), with an almost-sure infinite stopping time ($\tau = +\infty$).

**Theorem B.1.** *Under conditions (C.i)-(C.iii) above, there exists a unique pair* $(Y, Z) \in \mathscr{L}^{2,-2\alpha}(0,\infty;\mathbb{R}\times\mathbb{R}^d)$*, and with* $(Y, Z)$ *bounded, that satisfies BSDE (B.1).*

## B.2 Proof of Lemma 1

Suppose, leading to contradiction, that equilibrium exhibits non-zero risk prices on extrinsic shocks, i.e., $\pi_{n,t} \neq 0$ for at least one $n$.

The pricing Eq. (A.2), after integrating and using the no-bubble Condition 1, becomes the following present-value relation:

$$q_{n,t} = \mathbb{E}_t\Big[\int_t^\infty \frac{\xi_{n,s}}{\xi_{n,t}}\frac{y_{n,s}}{y_{n,t}} ds\Big].$$

With the assumption $\Gamma(\cdot) \equiv g$, and using Eqs. (A.14), (A.15), and (A.16), we have

$$q_{n,t} = \mathbb{E}_t\Big[\int_t^\infty \frac{G_{n,s}}{G_{n,t}} e^{-\int_t^s (r^* - \frac{\rho+1}{2\rho}\sum_{i=1}^N x_{i,u}\pi_{i,u}^2)du} ds\Big],$$

where $r^* := \delta + (\rho - 1)g - \frac{\rho(\rho-1)}{2}\nu^2$, and where

$$G_{n,t} = \exp\Big[-\int_0^t (\eta_u - \nu)dB_u + \int_0^t \hat{v}_{n,u}\cdot d\hat{B}_u - \int_0^t \pi_{n,u} dZ_u - \frac{1}{2}\int_0^t [(\eta_u - \nu)^2 + \|\hat{v}_{n,u}\|^2 + \pi_{n,u}^2]ds\Big]$$

is a martingale, hence can be used as a change-of-measure. Consequently,

$$q_{n,t} > \mathbb{E}_t\Big[\int_t^\infty \frac{G_{n,s}}{G_{n,t}} e^{-r^*(s-t)} ds\Big] = \mathbb{E}_t^{G_n}\Big[\int_t^\infty e^{-r^*(s-t)} ds\Big] = \frac{1}{r^*}.$$

Aggregating this result using $\alpha_{n,t} = y_{n,t}/Y_t$ as weights, we obtain the same inequality for the aggregate price-dividend ratio:

$$Q_t = \sum_{n=1}^{N} \alpha_{n,t} q_{n,t} > \frac{1}{r^*}. \tag{B.2}$$

On the other hand, let us examine agents' wealth-consumption ratios from Eq. (A.9). Substitute in consumption dynamics from (A.8), and use Eqs. (A.14), (A.15), and (A.16). The result is

$$\frac{w_{n,t}}{c_{n,t}} = \mathbb{E}_t\Big[\int_t^\infty \frac{J_{n,s}}{J_{n,t}} e^{-r^*(s-t)} e^{\frac{1-\rho}{2\rho^2}\int_t^s [\pi_{n,u}^2 + (1+\rho)\sum_{i=1}^N x_{i,u}\pi_{i,u}^2] du} ds\Big]$$

where

$$J_{n,t} = \exp\Big[(1-\rho)\nu B_t - \frac{1}{2}(1-\rho)^2\nu^2 t + \frac{1-\rho}{\rho}\int_0^t \pi_{n,u} dZ_{n,u} - \frac{1}{2}(\frac{1-\rho}{\rho})^2 \int_0^t \pi_{n,u}^2 du\Big]$$

is a martingale, hence can be used as a change-of-measure. Using the fact that $\rho \geq 1$, we thus have

$$\frac{w_{n,t}}{c_{n,t}} \leq \mathbb{E}_t\Big[\int_t^\infty \frac{J_{n,s}}{J_{n,t}} e^{-r^*(s-t)} ds\Big] = \mathbb{E}_t^{J_n}\Big[\int_t^\infty e^{-r^*(s-t)} ds\Big] = \frac{1}{r^*}.$$

Aggregating this result using $x_{n,t} = c_{n,t}/Y_t$ as weights, we obtain the same inequality for the aggregate wealth-consumption ratio, which is in fact equal to $Q_t$:

$$Q_t = \sum_{n=1}^{N} x_{n,t} \frac{w_{n,t}}{c_{n,t}} \leq \frac{1}{r^*}. \tag{B.3}$$

Eq. (B.3) contradicts Eq. (B.2), so we necessarily have $\pi_{n,t} = 0$ for all $n$ and $t$.

Now, given $\pi_{i,t} = 0$ for all $i$, we re-compute the location-specific price-dividend ratio:

$$q_{n,t} = \mathbb{E}_t\Big[\int_t^\infty \frac{G_{n,s}}{G_{n,t}} e^{-\int_t^s (r^* - \frac{\rho+1}{2\rho}\sum_{i=1}^N x_{i,u}\pi_{i,u}^2) du} ds\Big] = \mathbb{E}_t\Big[\int_t^\infty \frac{G_{n,s}}{G_{n,t}} e^{-r^*(s-t)} ds\Big] = \mathbb{E}_t^{G_n}\Big[\int_t^\infty e^{-r^*(s-t)} ds\Big] = \frac{1}{r^*}.$$

In particular, $\sigma_{n,t}^q = 0$ for all $n$.

## B.3 Proof of Lemma 2

Without loss of generality, we may set $\hat{\nu} = 0$ in this $N = 1$ case. We split the proof into the $\rho = 1$ case and the $\rho \neq 1$ case. In the case of log utility, $\rho = 1$, use Eq. (A.9) to obtain $w_t/c_t = Q_t = \delta^{-1}$, which has no volatility.

In the case of non-log utility, $\rho \neq 1$, we must proceed differently. Using the pricing Eq. (13), we can then write the dynamics of the aggregate price-dividend ratio $Q_t$ as

$$\frac{dQ_t}{Q_t} = \mu_t^Q dt + \varsigma_t^Q dB_t + \sigma_t^Q dZ_t,$$

where

$$\mu_t^Q = \delta - \frac{1}{2}\rho(\rho-1)\nu^2 + (\rho-1)\Gamma(Q_t) - \frac{1}{Q_t} + (\rho-1)\nu\varsigma_t^Q.$$

Define $Q^* := \{q : \delta - \frac{1}{2}\rho(\rho-1)\nu^2 + (\rho-1)\Gamma(q) = \frac{1}{q}\}$, which, as discussed in the text, exists uniquely. Then, let us rewrite the dynamics of $Q_t$ in a more canonical form in terms of $U_t := Q_t - Q^*$ with the Brownian shock vector $W := (B, Z)$:

$$\begin{aligned} dU_t &= -f(U_t, V_t)dt + V_t dW_t \\ f(u,v) &:= 1 - u\Big[\delta - \frac{1}{2}\rho(\rho-1)\nu^2 + (\rho-1)\Gamma(u)\Big] - (\rho-1)\nu v_1, \end{aligned}$$

where $v_1 := \binom{1}{0} \cdot v$. These dynamics constitute a 1-dimensional BSDE for $(U, V)$. One solution to this BSDE is clearly $(U, V) = 0$ (i.e., $Q_t = Q^*$ for all $t$).

To see that this is the only solution to the BSDE, we verify the assumptions (C.i)-(C.iii) directly preceding Theorem B.1. Since $f(0,0) = 0$, assumption (C.iii) is satisfied. Since $f(u,v)$ is continuously differentiable, it is Lipschitz and assumption (C.ii) is satisfied. Finally, compute

$$\begin{aligned} (u-u')[f(u,v) - f(u',v)] &= -(u-u')^2\Big(\delta - \frac{1}{2}\rho(\rho-1)\nu^2 + (\rho-1)\frac{u\Gamma(u) - u'\Gamma(u')}{u-u'}\Big) \\ &= -(u-u')^2\Big[\Delta(u) + (\rho-1)u'\frac{\Gamma(u)-\Gamma(u')}{u-u'}\Big] \end{aligned}$$

where $\Delta(u) := \delta - \frac{1}{2}\rho(\rho-1)\nu^2 + (\rho-1)\Gamma(u)$. Recall the assumption stated in the lemma that $\Delta(0) > 0$. Combining these assumptions with the facts that $\Gamma$ is increasing and $\rho \geq 1$, we therefore have

$$(u-u')[f(u,v) - f(u',v)] \leq -(u-u')^2\Delta(0).$$

Consequently, assumption (C.i) is satisfied with $\alpha = \Delta(0) > 0$. Theorem B.1 then implies a unique solution $(U,V) \in \mathscr{L}^{2,-2\Delta(0)}(0,\infty;\mathbb{R}\times\mathbb{R}^2)$. Given the solution $(U,V) = 0$ lies in this space, and any bounded pair $(U,V)$ would be a member of $\mathscr{L}^{2,-2\Delta(0)}(0,\infty;\mathbb{R}\times\mathbb{R}^2)$, we conclude that the BSDE has the unique bounded solution $(U,V) = 0$. As an unbounded solution would necessarily violate Condition 1, this concludes the proof.

## B.4 Proof of Lemma 3

The proof is identical to Lemma 2, applied location-by-location.

## B.5 Proof of Lemma 4

As stated, we assume throughout that we are in an equilibrium with $\sigma_{n,t}^q \neq 0$ for at least one $n$.

Suppose, leading to contradiction, that $\text{rank}(M) = N$. Then, by Eq. (A.17), we have $\pi_{n,t} = 0$ for all $n$. Using this fact, and using Eqs. (A.14), (A.15), and (A.16) in Eq. (A.8), consumption dynamics are

$$\frac{dc_{n,t}}{c_{n,t}} = g_t dt + \nu dB_t$$

Since these dynamics are independent of $n$, the wealth-consumption ratio in Eq. (A.9) is independent of $n$. Consequently, the exposure of wealth evolution $dw_{n,t}$ to the extrinsic shock $d\tilde{Z}_t$

must be independent of $n$. From dynamic budget constraint (3), along with equity market clearing $\theta_{n,t} = q_{n,t}y_{n,t}$, the exposure of $dw_{n,t}$ to $d\tilde{Z}_t$ is in fact $q_{n,t}y_{n,t}\sigma^q_{n,t}M_n$. Since $\sigma^q_{n,t} \neq 0$ for at least one $n$ (by assumption), then $q_{n,t}y_{n,t}\sigma^q_{n,t}M_n$ can only be independent of $n$ if all rows of $M$ are identical. But this contradicts $\text{rank}(M) = N$ for $N > 1$.

## B.6 Proof of Theorem 1

The proof consists of two steps: first, we construct a candidate equilibrium with non-fundamental volatility, using all equations in Appendix A; second, we verify that free disposal and No-Ponzi conditions hold, so that the construction is indeed an equilibrium.

**Step 1: Construct Candidate Equilibrium.** First, we construct a candidate equilibrium, assuming $(\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N$ are known. For some $\epsilon > 0$, define the stopping time $\tau := \inf\{t \geq 0 : \min_n q_{n,t} \leq \epsilon\} \wedge \inf\{t \geq 0 : \min_n x_{n,t} \leq \epsilon\}$. For $t < \tau$, let $\psi_t$ be an arbitrary non-zero adapted scalar process. For $t \geq \tau$, set $\psi_t = 0$.

Now, put $\pi_{n,t} = \delta\psi_t v^*_n / x_{n,t}$, where recall $v^* := (v^*_1, \ldots, v^*_N)' \neq 0$ is in the null-space of $M'$. Eq. (A.17), which says $M'p_t = 0$ for $p_t := (x_{1,t}\pi_{1,t}, \ldots, x_{N,t}\pi_{N,t})'$, holds because we have put $p_t = \delta\psi_t v^*$.

Next, set $(r_t, \eta_t, \hat{\eta}_t)$ according to (A.14), (A.15), and (A.16). Given $(\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N$ and $(\pi_{n,t})_{n=1}^N$ just constructed, this is feasible. At this point, we have all the dynamics (A.1) of the state price density, thus giving us $\xi_{n,t}/\xi_{n,0}$, so we may set $c_{n,t}/c_{n,0}$ according to (A.7). Note also that the individual transversality condition (A.5) holds, since $\mathbb{E}_t[\xi_{n,T}w_{n,T}] = \mathbb{E}_t[e^{-\delta T}\omega_{n,T}] = \delta\mathbb{E}_t[e^{-\delta T}] = \delta e^{-\delta T} \to 0$.

Next, consider the wealth-consumption ratios $\omega_{n,t} := w_{n,t}/c_{n,t}$. Since $\rho = 1$, note that $\omega_{n,t} = \delta^{-1}$, so that $\mu^\omega_{n,t} = 0$, $\varsigma^\omega_{n,t} = 0$, $\hat{\varsigma}^\omega_{n,t} = 0$, $\sigma^\omega_{n,t} = 0$, and Eqs. (A.6)-(A.10) hold. To obtain $c_{n,0}$ and thus the entire path of consumption, we use the initial condition $w_{n,0} = q_{n,0}y_{n,0}$, along with the wealth-consumption ratio. Indeed, $c_{n,0} = \delta w_{n,0} = \delta q_{n,0}y_{n,0}$.

Next, we set $\sigma^q_{n,t}$ by Eq. (A.22). Note that this proves the claim

$$\alpha_{n,t}q_{n,t}\sigma^q_{n,t} = \psi_t v^*_n$$

and that $\sigma^q_{n,t} \neq 0$ for some $n, t$ (i.e., this candidate equilibrium has self-fulfilling volatility). Put $(\varsigma^q_{n,t})_{n=2}^N$ and $(\hat{\varsigma}^q_{n,t})_{n=2}^N$ arbitrarily, and set $\varsigma^q_{1,t}$ and $\hat{\varsigma}^q_{1,t}$ to solve Eqs. (A.20) and (A.21), respectively. Given these objects constructed so far, we obtain portfolios $(\vartheta_{n,t}, \hat{\vartheta}_{n,t})$ via Eqs. (A.11)-(A.12). Again, these steps are feasible given $(\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N$, $(\pi_{n,t})_{n=1}^N$, and $\omega_{n,t} = \delta^{-1}$.

To then construct the path of the state variables $(\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N$, start with given initial condition $(\alpha_{n,0}, x_{n,0}, q_{n,0})_{n=1}^N > 0$ satisfying $\sum_{n=1}^N \alpha_{n,0} = \sum_{n=1}^N x_{n,0} = 1$ and $\sum_{n=1}^N \alpha_{n,0}q_{n,0} = \delta^{-1}$. Given the last condition, the aggregate wealth constraint (A.19) will automatically hold at $t = 0$. The dynamics of $(\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N$ are contained in Eqs. (A.2), (A.4), and (A.18). These dynamics are in closed-form, given the values of $(\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N$, because the previous steps in this equilibrium construction have now delivered every other relevant variable in terms of $(\alpha_{n,t}, x_{n,t}, q_{n,t})_{n=1}^N$. By inspection, one can also check that $\sum_{n=1}^N \alpha_{n,t} = \sum_{n=1}^N x_{n,t} = 1$ and $\sum_{n=1}^N \alpha_{n,t}q_{n,t} = \delta^{-1}$ will hold for all $t > 0$, given these dynamics. Thus, all equations in Appendix A hold for all $t \geq 0$.

**Step 2: Verify Free Disposal and No-Ponzi Conditions.** Note that free disposal automatically holds if $(q_n)_{n=1}^N \geq 0$, which is assumed in the statement of the theorem.

At this point, it remains to verify that the No-Ponzi conditions hold. We actually start by verifying the no-bubble condition of Condition 1:

$$\begin{aligned}
\lim_{T\to\infty} \mathbb{E}_t[\xi_{n,T} q_{n,T} y_{n,T}] &= \lim_{T\to\infty} \mathbb{E}_t[\alpha_{n,T} q_{n,T} e^{-\delta T} \frac{1}{x_{n,T}}] \\
&\leq \lim_{T\to\infty} \mathbb{E}_t[q_{n,T} e^{-\delta T} \frac{1}{x_{n,T}}] \\
&\leq K \lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} \frac{1}{x_{n,T}}] = 0.
\end{aligned}$$

In the first line, we have used (A.7); in the second line, we have used the fact that $\alpha_{n,T} \leq 1$; in the third line, we have used the boundedness of $q_n$ by some $K$, and then the theorem's stated assumption that $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} \frac{1}{x_{n,T}}] = 0$. This proves that Condition 1 holds.

Next, note that $w_{n,t} = \delta^{-1} c_{n,t} = \delta^{-1} x_{n,t} Y_t$, so that $w_{n,t} \geq 0$ if and only if $x_{n,t} \geq 0$. The latter inequality is proved by inspecting the dynamics (A.18) and noting that the definition of the stopping time $\tau$ ensures that $x_{n,t} \geq 0$ almost-surely.

Now, since $w_{n,t}$ and $q_{n,t}$ are both positive, and since $\xi_{n,t}$ is the local state-price density, we know $(\xi_{n,t} w_{n,t})_{t\geq 0}$ and $(\xi_{n,t} y_{n,t} q_{n,t})_{t\geq 0}$ are both continuous, positive super-martingales. So by Doob's super-martingale convergence theorem, we know that $\lim_{T\to\infty} \xi_{n,T} w_{n,T}$ and $\lim_{T\to\infty} \xi_{n,T} y_{n,T} q_{n,T}$ both exist and are finite. Next, transversality condition (A.5) and no-bubble Condition 1 imply there exists a sub-sequence of times $\{T_j\}_{j=1}^{\infty}$ along which $\lim_{j\to\infty} \xi_{n,T_j} w_{n,T_j} = 0$ and $\lim_{j\to\infty} \xi_{n,T_j} y_{n,T_j} q_{n,T_j} = 0$. But these limits must be the same along any subsequence, by the first step (i.e., that the limits exist), which shows $\lim_{T\to\infty} \xi_{n,T} w_{n,T} = \lim_{T\to\infty} \xi_{n,T} y_{n,T} q_{n,T} = 0$. Finally, combine the previous limits with equity market clearing $\theta_{n,T} = q_{n,T} y_{n,T}$ to obtain (4).

## B.7 Proof of Proposition 1

Consider $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$ with $\lambda > \delta^2$ and fixed $\epsilon$ that satisfies $0 < \epsilon < \delta^{-2} - \lambda^{-1}$. Also note that, given $\rho = 1$, Eq. (A.9) implies that all wealth-consumption ratios are constant over time and across locations at $\omega_{n,t} = \delta^{-1}$. The general proof strategy will be to conjecture asset price processes that feature extrinsic volatility and then verify the conditions of Theorem 1 (roughly speaking, boundedness of valuations and survival of all agents), so that the conjectured dynamics are consistent with equilibrium.

Supposing $\mathrm{rank}(M) < N$, let $v^* := (v_1^*, \ldots, v_N^*)$ be in the null-space of $M'$, and conjecture a stochastic equilibrium exists with $\alpha_{n,t} q_{n,t} \sigma_{n,t}^q = v_n^* \psi_t$ for some process $\psi_t$. By Eq. (A.22), this conjecture implies $\pi_{n,t} = \delta v_n^* \psi_t / x_{n,t}$. We also conjecture an equilibrium with $\varsigma_{n,t}^q = 0$, which satisfies Eq. (A.20). Finally, conjecture an equilibrium with $\hat{\varsigma}_{n,t}^q = 0$ if $n \neq n_t^* := 1 + \arg\min_n q_{n,t}$. To satisfy Eq. (A.21), we must set

$$\hat{\varsigma}_{n_t^*,t}^q = -\sum_{n=1}^{N} \frac{q_{n,t} \alpha_{n,t}}{q_{n_t^*,t} \alpha_{n_t^*,t}} \hat{v}_{n,t}.$$

Under these conjectures, we will use properties (P1) and (P2) in Proposition 1 to verify the conditions of Theorem 1.

Define

$$D(q) := -1 + (\delta + \lambda\delta^{-1})q - \lambda q^2. \tag{B.4}$$

Note that $D(q) = 0$ is a quadratic equation that has two roots: $\delta^{-1}$ and $\delta\lambda^{-1}$. Moreover, $D(q) > 0$ if and only if $q \in (\delta\lambda^{-1}, \delta^{-1})$. Substituting the above conjectures and all other equilibrium objects into the asset-pricing Eq. (A.2), we have

$$(\text{if } n \neq n_t^*) \quad dq_{n,t} = \Big[D(q_{n,t}) - \Big(\delta^2 \psi_t^2 \sum_{i=1}^N \frac{(v_i^*)^2}{x_{i,t}}\Big) q_{n,t} + \delta \frac{(v_n^* \psi_t)^2}{\alpha_{n,t} x_{n,t}}\Big] dt + \frac{v_n^*}{\alpha_{n,t}} \psi_t dZ_{n,t} \tag{B.5}$$

$$(\text{if } n = n_t^*) \quad dq_{n,t} = \Big[D(q_{n,t}) - \Big(\delta^2 \psi_t^2 \sum_{i=1}^N \frac{(v_i^*)^2}{x_{i,t}}\Big) q_{n,t} + \delta \frac{(v_n^* \psi_t)^2}{\alpha_{n,t} x_{n,t}} - \hat{v}_{n,t} \cdot \hat{\varsigma}_{n,t}^q\Big] dt + \frac{v_n^*}{\alpha_{n,t}} \psi_t dZ_{n,t} + q_{n,t} \hat{\varsigma}_{n,t}^q d\hat{B}_t \tag{B.6}$$

An important fact to observe is the following: under the assumptions of the proposition, the dynamics of $\underline{q}_t := \min_n q_{n,t}$ and $\bar{q}_t := \max_n q_{n,t}$ both take the form of Eq. (B.5). Indeed, if $N \geq 3$, then $\arg\min_n q_{n,t} < n_t^* < \arg\max_n q_{n,t}$ by definition of $n_t^*$, so that Eq. (B.5) applies to $d\underline{q}_t$ and $d\bar{q}_t$. On the other hand, if $\hat{v} = 0$, then Eqs. (B.5) and (B.6) are equivalent, so again (B.5) applies to $d\underline{q}_t$ and $d\bar{q}_t$.

We now show that if properties (P1) and (P2) are satisfied, then $q_{n,t}$ remains bounded for all $n$. Under property (P2), we have $\psi_t = 0$ if $\underline{q}_t = \delta(\epsilon + \lambda^{-1})$, and so

$$\text{if } \underline{q}_t = \delta(\epsilon + \lambda^{-1}), \quad d\underline{q}_t = D\big(\delta(\epsilon + \lambda^{-1})\big) dt > 0.$$

Therefore, $\underline{q}_t$ can never cross $\delta(\epsilon + \lambda^{-1})$ from above in a path-continuous way. Under property (P1), the drift and diffusion coefficients of $\underline{q}_t$ are bounded, so $\underline{q}_t$ is almost-surely path-continuous. This proves that the entire path is bounded below: if $\underline{q}_0 \geq \delta(\epsilon + \lambda^{-1})$, then $\underline{q}_t \geq \delta(\epsilon + \lambda^{-1})$ for all $t$ almost-surely. An analogous argument applies to $\bar{q}_t$: properties (P1) and (P2) imply $\bar{q}_t$ can never cross $K\delta^{-1}$ from below. Thus, if $\bar{q}_0 \leq K\delta^{-1}$, then $\bar{q}_t \leq K\delta^{-1}$ for all $t$, almost-surely. Since $\underline{q}_t \leq q_{n,t} \leq \bar{q}_t$ for all $n$, we have proved the boundedness of $q_{n,t}$.

In summary, $\{(q_{n,t})_{n=1}^N : t \geq 0\}$ is positive and bounded almost-surely, so to verify the conditions of Theorem 1, it remains to show that $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} x_{n,T}^{-1}] = 0$. Substituting equilibrium objects into (A.18), we have

$$dx_{n,t} = \psi_t^2 \delta^2 \Big[(1 - x_{n,t}) \frac{(v_n^*)^2}{x_{n,t}} - x_{n,t} \sum_{i \neq n} \frac{(v_i^*)^2}{x_{i,t}}\Big] dt + \psi_t \delta v_n^* dZ_{n,t}. \tag{B.7}$$

Decompose $x_{n,T}$ into two parts as follows. Define

$$x_{n,T}^{\psi} := x_{n,0} \mathbf{1}_{\{\psi_0 > 0\}} + \int_0^T \mathbf{1}_{\{\psi_t > 0\}} dx_{n,t} \quad \text{and} \quad x_{n,T}^0 := x_{n,0} \mathbf{1}_{\{\psi_0 = 0\}} + \int_0^T \mathbf{1}_{\{\psi_t = 0\}} dx_{n,t}.$$

Clearly, $x_{n,T}^{\psi} + x_{n,T}^0 = x_{n,T}$. From Eq. (B.7), notice that $dx_{n,T}^0 = \mathbf{1}_{\{\psi_T = 0\}} dx_{n,T} = 0$, so that $(x_{n,T})^{-1} = (x_{n,0})^{-1} + (x_{n,T}^{\psi})^{-1} - (x_{n,0}^{\psi})^{-1}$. Putting these pieces together, we have

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} (x_{n,T})^{-1}] = \lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} \Big((x_{n,0})^{-1} + (x_{n,T}^{\psi})^{-1} - (x_{n,0}^{\psi})^{-1}\Big)] = \lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} (x_{n,T}^{\psi})^{-1}]$$

Finally, since $\psi_t / x_{n,t} \leq \psi_t / \min_i x_{i,t}$ is bounded, by requirement (P1), we have $(x_{n,t}^{\psi})^{-1}$ bounded, which proves that $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} (x_{n,T}^{\psi})^{-1}] = 0$.

## B.8 Proof of Proposition 2

To derive equilibrium with complete financial markets, we allow investor $n$ to take a position $\theta_{n,t}^*$ in an integrated futures market tracking the extrinsic shock $d\tilde{Z}_t$. The payoff of such a futures contract is $\pi_t dt + d\tilde{Z}_t$, where $\pi_t$ is the extrinsic risk-price vector (i.e., $\pi_{n,t} = \pi_t$ for all $n$ in a complete-markets setting). This futures market is in zero net supply, so $\sum_{n=1}^{N} \theta^*_{n,t} = 0$.

In this setting, we re-derive the equations in Appendix A. Most of the equations are the same, with the exception of Eqs. (A.13), (A.17), and (A.22). Eq. (A.13) now reads

$$\sigma^{\omega}_{n,t} = \frac{\theta^*_{n,t}}{w_{n,t}} + \frac{\theta_{n,t}}{w_{n,t}}\sigma^q_{n,t} - \rho^{-1}\pi_t \tag{B.8}$$

Eq. (A.17) still technically holds, but given $\pi_{n,t} = \pi_t$ for all $n$, it specializes to the degenerate solution

$$\pi_t = 0. \tag{B.9}$$

A particular implication of Eq. (B.9) is that the unique SDF takes the form $d\xi_t = -\xi_t[r_t dt + \eta_t dB_t]$, as stated in the text. The aggregate risk price and risk-free rate are now given by $\eta_t = \rho\nu$ and $r_t = \delta + \rho g_t - \frac{1}{2}\rho(\rho+1)\nu^2$. Therefore, equilibrium consumption dynamics are

$$\frac{dc_{n,t}}{c_{n,t}} = g_t dt + \nu dB_t,$$

which reflects perfect risk-sharing and is invariant to all idiosyncratic shocks (both $\hat{B}$ and $\tilde{Z}$). Finally, Eq. (A.22) must now be replaced by the following, which results by summing (B.8) over $n$ and using (B.9) as well as the equity and futures market clearing conditions $\theta_{n,t} = q_{n,t}y_{n,t}$ and $\sum_{n=1}^{N}\theta^*_{n,t} = 0$, respectively:

$$\sum_{n=1}^{N} w_{n,t}\sigma^{\omega}_{n,t} = \sum_{n=1}^{N} q_{n,t}y_{n,t}\sigma^q_{n,t}. \tag{B.10}$$

Using the results above, we may price the local equity as

$$\begin{aligned} q_{n,t} &= \mathbb{E}_t\Big[\int_t^\infty \frac{\xi_s}{\xi_t}\frac{y_{n,s}}{y_{n,t}} ds\Big] \\ &= \mathbb{E}_t\Big[\int_t^\infty \frac{G_{n,s}}{G_{n,t}} \exp\Big(-\int_t^s [r_u - g_{n,u} + \nu\eta_u]du\Big) ds\Big] \end{aligned}$$

where

$$G_{n,t} := \exp\Big[-\frac{1}{2}\int_t^s (\nu^2 + \eta_u^2 - 2\nu\eta_u)du + \int_t^s (\nu - \eta_u)dB_u - \int_t^s \frac{1}{2}\|\hat{v}_{n,u}\|^2 du + \int_t^s \hat{v}_{n,u}\cdot d\hat{B}_u\Big].$$

Given $G_{n,t}$ is an exponential local martingale, we can use it to change measure from $\mathbb{E}$ to $\mathbb{E}^{G_n}$, so

$$q_{n,t} = \mathbb{E}_t^{G_n}\Big[\int_t^\infty \exp\Big(-\int_t^s [r_u - g_{n,u} + \nu\eta_u]du\Big) ds\Big].$$

These equations so far hold generally for any $\rho$, any $\hat{v}$, and any function $\Gamma(\cdot)$.

Finally, we follow the arguments of Proposition 1 identically to show that $q_{n,t}$ is bounded and that $\lim_{T\to\infty}\mathbb{E}_t[e^{-\delta T}x_{n,t}^{-1}] = 0$. The only minor difference is that $\pi_{n,t} = \pi_t = 0$, so that

$$\begin{aligned} &(\text{if } n \neq n^*_t) \quad dq_{n,t} = D(q_{n,t})dt + \frac{v^*_n}{\alpha_{n,t}}\psi_t dZ_{n,t} \\ &(\text{if } n = n^*_t) \quad dq_{n,t} = \Big[D(q_{n,t}) - \hat{v}_{n,t}\cdot\hat{\varsigma}^q_{n,t}\Big]dt + \frac{v^*_n}{\alpha_{n,t}}\psi_t dZ_{n,t} + q_{n,t}\hat{\varsigma}^q_{n,t}d\hat{B}_t. \end{aligned}$$

With this modification, the arguments go through identically, so we omit them here.

# Internet Appendix

(Not for publication)

# Segmentation and Beliefs:

## A Theory of Self-Fulfilling Idiosyncratic Risk

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## C Other stabilizing forces

This online appendix provides three additional microfoundations for sources of endogeneity that keep valuation ratios stable—therefore, we call these *stabilizing forces*. In Section C.1, we replace the valuation-growth link with a connection between valuations and beliefs about growth. In Section C.2, we model firms that invest, subject to a debt overhang problem, which microfounds connection between valuations and growth—this is similar to our baseline model but with a particular microfoundation. In Section C.3, we model a creative destruction process that depends on valuations. In all of the extensions in this appendix, we will assume that agents have log utility ($\rho = 1$).

### C.1 Valuation-dependent beliefs as a "stabilizing force"

In the main text, we study a positive connection between asset valuations and growth. Here, we explore a model in which asset valuations increase *beliefs about growth* rather than actual growth. For reasons that will become clear, self-fulfilling volatility requires segmented futures markets (i.e., no cross-location trading on the $dB_t$ shock); if futures markets were integrated, all agents would agree on the aggregate risk price, and beliefs would not affect asset valuations. Unfortunately, the analysis of this setting is substantially more complex than our baseline model, so we will specialize to an economy with constant true growth rates $g$, without any idiosyncratic risk ($\hat{v} = 0$), and with an additional cross-location entry/exit margin that facilitates analysis of the wealth distribution. More details on this entry/exit margin below. Furthermore, we will eventually specialize to a two-location economy, in which one location is vanishingly small (like a small open economy).

**Endowments.** Each location receives identical geometric Brownian motions

$$\frac{dy_{n,t}}{y_{n,t}} = g dt + v dB_t$$

Therefore, the aggregate output also follows $dY_t/Y_t = g dt + v dB_t$. Furthermore, each locations' endowment share is constant over time. Therefore, we write $\alpha_n$ for the location-$n$ endowment share, dropping the time subscript.

**Beliefs.** Let $\mathbb{P}$ be the objective probability measure. Subjective beliefs are modeled as follows. For some process $\gamma_{n,t}$, we define the likelihood ratio between subjective beliefs and the physical probability as

$$H_{n,t} := \Big(\frac{d\widetilde{\mathbb{P}}^n}{d\mathbb{P}}\Big)_t = \exp\Big[\int_0^t \gamma_{n,s} dB_s - \frac{1}{2}\int_0^t \gamma_{n,s}^2 ds\Big]. \tag{C.1}$$

By Girsanov's theorem, this amounts to assuming that agents in location $n$ believe that $d\tilde{B}_{n,t} := dB_t - \gamma_{n,t}dt$ is a Brownian motion. Meanwhile, agents have rational beliefs about all other shocks. As with the endogeneity in fundamental growth rates from Proposition 1, we assume that

$$\gamma_{n,t} = \frac{\lambda}{\nu}(q_{n,t} - \delta^{-1}), \quad \lambda > 0. \tag{C.2}$$

Equation (C.2) says that investors become more optimistic about growth when prices rise. An implication of these assumptions is that agent $n$ holds the following subjective belief $\tilde{g}_{n,t} := \frac{1}{dt}\widetilde{\mathbb{E}}_t^n[\frac{dy_{n,t}}{y_{n,t}}]$ about his local endowment growth rate:

$$\tilde{g}_{n,t} = g + \lambda(q_{n,t} - \delta^{-1}). \tag{C.3}$$

This mirrors Eq. (19), but for perceived growth rather than true growth.

**Valuations.** In general, as there are no $d\hat{B}_t$ shocks, asset valuations take the form

$$\frac{dq_{n,t}}{q_{n,t}} = \mu_{n,t}^q dt + \varsigma_{n,t}^q dB_t + \sigma_{n,t}^q dZ_{n,t}$$

However, we will conjecture an equilibrium in which $\varsigma_{n,t}^q = 0$ for all $n$.

**Optimization and risk prices.** Without hedging markets for the aggregate $dB_t$ shock, location $n$ has its own aggregate risk price, and its SDF follows

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \eta_{n,t}dB_t + \pi_{n,t}dZ_{n,t}\Big].$$

Different to the baseline model, marginal utility incorporates the belief distortion, so optimal consumption sets

$$H_{n,t}e^{-\delta t}c_{n,t}^{-1} = \xi_{n,t}.$$

Thus, optimal consumption dynamics for each location $n$ are then

$$\frac{dc_{n,t}}{c_{n,t}} = \Big[r_t - \delta - \gamma_{n,t}(\gamma_{n,t} + \eta_{n,t}) + (\gamma_{n,t} + \eta_{n,t})^2 + \pi_{n,t}^2\Big]dt + (\gamma_{n,t} + \eta_{n,t})dB_t + \pi_{n,t}dZ_{n,t}. \tag{C.4}$$

As before, with log utility, the location-$n$ wealth-consumption ratio is equal to $\omega_{n,t} := \frac{w_{n,t}}{c_{n,t}} = \delta^{-1}$. Apply Itô's formula to this result, using the dynamic budget constraint (3) with the following substitutions: $\vartheta_{n,t} = 0$ (since there are no futures markets), $\eta_t$ replaced by the location-specific risk price $\eta_{n,t}$ (again, since there are no futures markets), $\theta_{n,t} = q_{n,t}y_{n,t}$ (equity market clearing), and imposing the conjecture $\varsigma_{n,t}^q = 0$. The results are

$$\eta_{n,t} + \gamma_{n,t} = \frac{\delta\alpha_n q_{n,t}}{x_{n,t}}\nu \tag{C.5}$$

$$\pi_{n,t} = \frac{\delta\alpha_n q_{n,t}}{x_{n,t}}\sigma_{n,t}^q. \tag{C.6}$$

In other words, the risk exposures of representative agent $n$ coincide with the risks they hold through their local equity.

**Aggregation.** Applying Itô's formula to the goods market clearing condition $\sum_{n=1}^{N} c_{n,t} = Y_t$, we obtain

$$r_t = \delta + g + \sum_{n=1}^{N} x_{n,t}\gamma_{n,t}(\gamma_{n,t} + \eta_{n,t}) - \sum_{n=1}^{N} x_{n,t}[(\gamma_{n,t} + \eta_{n,t})^2 + \pi_{n,t}^2] \tag{C.7}$$

from matching drifts, and

$$\sum_{n=1}^{N} \alpha_n q_{n,t} = \delta^{-1} \tag{C.8}$$

$$\sum_{n=1}^{N} \alpha_n q_{n,t} \sigma_{n,t}^q M_n = 0 \tag{C.9}$$

from matching diffusion coefficients and substituting Eqs. (C.5)-(C.6) above for $\eta_{n,t}$ and $\pi_{n,t}$. Eq. (C.8) is simply the aggregate wealth constraint. Eq. (C.9) is a constraint on the relative extrinsic volatilities. To satisfy this constraint, we let $M$ be any matrix with rank$(M) < N$ and let $v^*$ be a vector in the null-space of $M'$. Then, introduce a positive process $\psi_t$ (as in Proposition 1) and let

$$\alpha_n q_{n,t} \sigma_{n,t}^q = \psi_t v_n^*. \tag{C.10}$$

Clearly, Eq. (C.10) solves Eq. (C.9), since $M'v^* = 0$. The dynamics of $x_{n,t} = c_{n,t}/Y_t$ are given by applying Itô's formula to its definition:

$$\begin{aligned} \frac{dx_{n,t}}{x_{n,t}} = &\Big[r_t - \delta - g - \gamma_{n,t}(\gamma_{n,t} + \eta_{n,t}) - \nu(\gamma_{n,t} + \eta_{n,t}) + \nu^2 + (\gamma_{n,t} + \eta_{n,t})^2 + \pi_{n,t}^2\Big]dt \\ &+ (\gamma_{n,t} + \eta_{n,t} - \nu)dB_t + \pi_{n,t} dZ_{n,t}. \end{aligned} \tag{C.11}$$

Finally, the equilibrium asset-pricing condition is

$$\mu_{n,t}^q + g + \frac{1}{q_{n,t}} - r_t = \nu\eta_{n,t} + \sigma_{n,t}^q \pi_{n,t}. \tag{C.12}$$

This completes the set of equilibrium equations, analogous to Appendix A. The key question, building off of Theorem 1, is whether the dynamics above induce stationary valuations $(q_{n,t})_{n=1}^N$ and stationary consumption shares $(x_{n,t})_{n=1}^N$.

**Entry/exit margin.** We assume in reduced-form that entry/exit occurs between the locations in a way that keeps $\eta_{n,t} + \gamma_{n,t} \leq \bar{\eta}$ for all $n, t$. Such an assumption is reasonable, because the Sharpe ratios represent risk-adjusted profits to investors. In fact, with log utility, with an entry cost that is proportional to wealth, and in an equilibrium without self-fulfilling volatility, this is actually the optimal entry process, as shown in Khorrami (2018). Different entry costs map into different values of $\bar{\eta}$. We will assume $\bar{\eta} > \nu$, i.e., entry occurs when Sharpe ratios are somewhat above the perfect risk-sharing Sharpe ratio. Using Eq. (C.5), such an entry process translates into a lower bound for $x_{n,t}$:

$$x_{n,t} \geq \underline{x}_{n,t} := \bar{\eta}^{-1}\delta\alpha_n q_{n,t}\nu. \tag{C.13}$$

When $x_{n,t}$ falls, Sharpe ratios rise, which provides an incentive for investors to flow from other locations into location $n$, keeping $x_{n,t} \geq \underline{x}_{n,t}$. Thus, $\underline{x}_{n,t}$ is a reflecting boundary for $x_{n,t}$. Modeling entry in this way substantially simplifies the analysis of the equilibrium dynamical system.

**Steady state.** The equilibrium dynamical system for $(x_{n,t}, q_{n,t})_{n=1}^N$ is governed by Eqs. (C.12) and (C.11). If there is no self-fulfilling volatility, $\psi_t = 0$, then this dynamical system has a deterministic steady state which is given by $x_{n,t} = \alpha_n$ and $q_{n,t} = \delta^{-1}$ for all $n$. Although the stability properties of the dynamical system are much more complicated in this model than in our baseline model, by specializing to $N = 2$ locations and treating one location as "small", we may obtain some sharp analytical results.

**Example with one small and one large location.** To transparently establish the existence of a sunspot equilibrium, we now specialize to $N = 2$ locations. With $N = 2$, we can focus on location-1 and determine the location-2 equilibrium objects via market clearing. In particular, drop the location subscripts and denote $\alpha := \alpha_1$, $x_t := x_{1,t}$, and $q_t := q_{1,t}$. Then, the location-2 objects are $\alpha_2 = 1 - \alpha$, $x_{2,t} = 1 - x_t$, and

$$q_{2,t} = \frac{\delta^{-1} - \alpha q_t}{1 - \alpha}$$

Furthermore, we will assume (as in Example 1) that

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad \text{so that} \quad v^* = (1,1)' \in \text{nullspace}(M').$$

This specification is equivalent to assuming $dZ_{2,t} = -dZ_{1,t}$. Therefore, let us define the single extrinsic shock $Z_t := Z_{1,t}$.

Let us focus now on the location-1 valuation $q_t$ and consumption share $x_t$. Substitute Eqs. (C.2), (C.5), (C.6), and (C.10) into Eq. (C.12) to obtain

$$dq_t = \Big[ -1 + \frac{\delta \psi_t^2}{\alpha x_t} + \Big(r_t - g + \lambda\delta^{-1}\Big) q_t - \Big(\lambda - \frac{\delta\alpha}{x_t}\nu^2\Big) q_t^2 \Big] dt + \frac{\psi_t}{\alpha} dZ_t. \tag{C.14}$$

Then, substituting (C.7) into (C.14) and doing some algebra, we obtain

$$dq_t = \Big[ -1 + \Big(\frac{\delta}{\alpha x_t} - \frac{\delta^2 q_t}{x_t(1 - x_t)}\Big)\psi_t^2 + A_{1,t} q_t + A_{2,t} q_t^2 + A_{3,t} q_t^3 \Big] dt + \frac{\psi_t}{\alpha} dZ_t \tag{C.15}$$

$$\begin{aligned} \text{where} \quad A_{1,t} &:= \delta + \frac{\lambda\delta^{-1}}{1 - \alpha} - \frac{\nu^2}{1 - x_t} \\ A_{2,t} &:= \alpha\Big(\frac{\delta\nu^2}{x_t} + \frac{2\delta\nu^2}{1 - x_t} - \frac{2\lambda}{1 - \alpha}\Big) - \lambda \\ A_{3,t} &:= \alpha\Big(\frac{\lambda\delta}{1 - \alpha} - \frac{\alpha(\delta\nu)^2}{x_t(1 - x_t)}\Big) \end{aligned}$$

Similarly, substitute various results into Eq. (C.11), we obtain

$$\begin{aligned} dx_t = \Big[ & x_t \frac{\lambda\delta^{-1}\alpha}{1 - \alpha}(1 - \delta q_t)^2 - \lambda(q_t - \delta^{-1})\alpha\delta q_t - \alpha\delta q_t \nu^2 \\ & + \frac{(\alpha\delta q_t)^2}{x_t}\nu^2 - \frac{(x_t - \alpha\delta q_t)^2}{1 - x_t}\nu^2 + \frac{(\delta\psi_t)^2}{x_t} - \frac{(\delta\psi_t)^2}{1 - x_t}\Big] dt + (\delta\alpha q_t - x_t)\nu dB_t + \delta\psi_t dZ_t. \end{aligned} \tag{C.16}$$

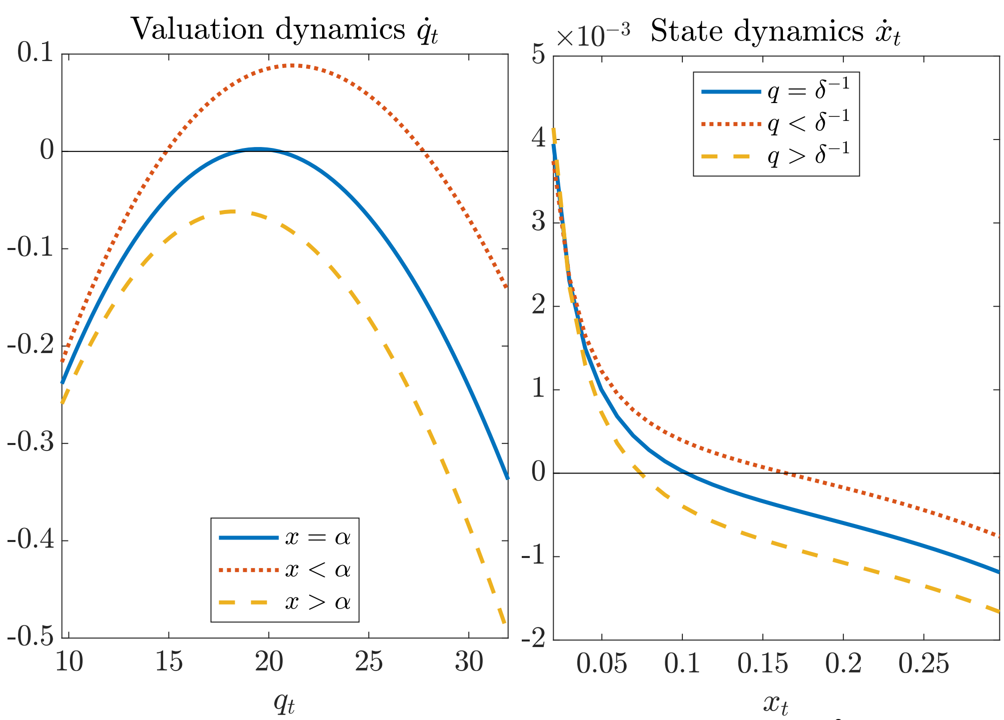
Given the entry process, consumption shares also obey $x_t \geq \bar{\eta}^{-1}\delta\nu\alpha q_t$ and $1 - x_t \geq \bar{\eta}^{-1}\delta\nu(1 - \alpha)q_{2,t}$. Combining these bounds and using the expression for $q_{2,t}$, equilibrium has

$$\bar{\eta}^{-1}\delta\nu\alpha q_t \leq x_t \leq 1 + \bar{\eta}^{-1}\delta\nu\alpha q_t - \bar{\eta}^{-1}\nu. \tag{C.17}$$

Equilibrium requires the dynamics (C.15) to be such that $q_t > 0$ and $q_t < \delta^{-1}/\alpha$ (so that $q_{2,t} > 0$) for all $t$.

Figure C.1 provides an illustration of the drifts of Eqs. (C.15) and (C.16) when $\psi_t = 0$. The dynamics look like they could be locally stable (see the solid and dotted lines in the left panel, near the higher steady state), but this conclusion seems to depend on the level of $x_t$. Of course, this figure also depends on a specific choice of parameters. So the question is whether some more general statements can be made about dynamical stability.

Figure C.1: Valuation and consumption share dynamics.




*Notes.* Parameters are $\delta = 0.05$, $g = 0.02$, $\nu = 0.1$, $\alpha = 0.1$, $\lambda = \frac{\delta^2}{1-\delta} + \delta\nu^2$.

Proving the general stationarity of $(q_t)_{t\geq 0}$ is technically difficult, so we sketch the main ideas in a limiting case in which one location is vanishingly small. This is essentially a "small open economy" limit. In particular, for each $\alpha$, the equilibrium is indexed as follows. Let $\psi_t = \alpha\psi_t^*$ be the self-fulfilling volatility process (this intentionally vanishes with $\alpha$). Let $x_t^\alpha$ and $q_t^\alpha$ be the resulting consumption share and valuation in equilibrium. Thus, $(x_t, q_t, \psi_t)_{t\geq 0} = (x_t^\alpha, q_t^\alpha, \alpha\psi_t^*)_{t\geq 0}$ is the equilibrium for a fixed $\alpha$. We will take $\alpha \to 0$ and establish the desired stability properties in that limiting equilibrium. Let $x_t^* := \lim_{\alpha\to 0} x_t^\alpha$ and $q_t^* := \lim_{\alpha\to 0} q_t^\alpha$ be the limiting equilibrium objects.

In this limiting equilibrium, $x_t^* = 0$ with probability 1. Indeed, inspecting the dynamics (C.16) with $\alpha \to 0$ and $\psi_t = \alpha\psi_t^* \to 0$, we see that

$$dx_t^* = -(x_t^*\nu)^2 dt - x_t^*\nu dB_t.$$

The initial consumption share of location 1 is $x_0^\alpha = \alpha\delta q_0^\alpha$, so $x_0^* = 0$. Using the dynamics above, we then have $x_t^* = 0$ for all $t$.

Define $\tilde{x}_t^* := \lim_{\alpha \to 0} x_t^\alpha / \alpha$, and note its initial value $\tilde{x}_0^* = \delta q_0^*$. Given the entry/exit margin, captured in Eq. (C.13), we have $\tilde{x}_t^* \geq \bar{\eta}^{-1} \delta \nu q_t^*$ (the upper bound scaled by $1/\alpha$ diverges and becomes irrelevant as $\alpha$ shrinks).

We can now examine the limiting dynamics for $q_t^*$ and $\tilde{x}_t^*$:

$$dq_t^* = \Big[ -1 + \frac{\delta}{\tilde{x}_t^*} (\psi_t^*)^2 + \Big( \delta + \lambda \delta^{-1} - \nu^2 \Big) q_t^* + \Big( \frac{\delta \nu^2}{\tilde{x}_t^*} - \lambda \Big) (q_t^*)^2 \Big] dt + \psi_t^* dZ_t \tag{C.18}$$

$$d\tilde{x}_t^* = \Big[ \frac{(\delta q_t^*)^2}{\tilde{x}_t^*} \nu^2 - \delta q_t^* [\nu^2 + \lambda (q_t^* - \delta^{-1})] + \frac{(\delta \psi_t^*)^2}{\tilde{x}_t^*} \Big] dt + (\delta q_t^* - \tilde{x}_t^*) \nu dB_t + \delta \psi_t^* dZ_t \tag{C.19}$$

$$\text{where} \quad \tilde{x}_t^* \geq \bar{\eta}^{-1} \delta \nu q_t^*. \tag{C.20}$$

A steady state of this system is $(\tilde{x}_t^*, q_t^*, \psi_t^*) = (1, \delta^{-1}, 0)$. To show that self-fulfilling volatility is possible (i.e., $\psi_t^* \neq 0$), we need to show that $q_t^* > 0$ for all $t$ with probability 1. To do this, we need the following parameter restrictions:

$$\delta > \nu^2 \tag{C.21}$$

$$\lambda > \delta^2 + \delta \nu^2 + 2 \nu \delta^{1.5} \tag{C.22}$$

$$\nu < \bar{\eta} < \frac{1}{2} \frac{\delta + \lambda \delta^{-1} - \nu^2}{\nu} \tag{C.23}$$

Note that (C.21)-(C.23) are mutually consistent (i.e., the proposed interval for $\bar{\eta}$ is non-empty).

Consider the first-passage time

$$\tau := \Big\{ t \geq 0 : q_t^* \leq \underline{q}_t^* := \frac{\delta + \lambda \delta^{-1} - \nu^2}{2(\lambda - \delta \nu^2 / \tilde{x}_t^*)} \Big\}. \tag{C.24}$$

Let $\psi_\tau^* = 0$, so that self-fulfilling volatility vanishes as valuations reach the lower bound specified in (C.24). Then, we have the following lemma, which shows that the equilibrium is stable and therefore permits self-fulfilling volatility.

**Lemma C.1.** *Under parameter assumptions (C.21)-(C.23), we have $dq_\tau^* > 0$ almost-surely, and consequently $(q_t^*)_{t \geq 0} > 0$ given any process $(\psi_t^*)_{t \geq 0}$ that vanishes as $q_t^*$ approaches $\underline{q}_t^*$.*

*Proof.* We will first conjecture and then verify that $\tilde{x}_t^* > \delta \nu^2 / \lambda$. Given this conjecture, notice from the definition of $\underline{q}_t^*$ in (C.24) that

$$\underline{q}_t^* > \frac{\delta + \lambda \delta^{-1} - \nu^2}{2\lambda}. \tag{C.25}$$

Under parameter assumption (C.21), the right-hand-side of the expression above is strictly positive. Combine parameter assumption (C.23) with the entry barrier in (C.20), along with $q_t^* \geq \underline{q}_t^*$ and the lower bound for $\underline{q}_t^*$ in (C.25). The result is that we verify

$$\tilde{x}_t^* > \frac{\delta \nu^2}{\lambda}. \tag{C.26}$$

Next, we need to show that $dq_\tau^* > 0$ if $\psi_\tau^* = 0$. Consider the function

$$f(q; x) := -1 + \Big( \delta + \lambda \delta^{-1} - \nu^2 \Big) q + \Big( \frac{\delta \nu^2}{x} - \lambda \Big) q^2$$

Note that $dq_\tau^* = f(q_\tau^*; \tilde{x}_\tau^*)dt$. As a function of $q$, $f(q;x)$ is a quadratic function with two roots $q_+$ and $q_-$, which are

$$q_+(x) = \frac{\delta + \lambda\delta^{-1} - \nu^2 + \sqrt{(\delta + \lambda\delta^{-1} - \nu^2)^2 - 4(\lambda - \frac{\delta\nu^2}{x})}}{2(\lambda - \frac{\delta\nu^2}{x})}$$
$$q_-(x) = \frac{\delta + \lambda\delta^{-1} - \nu^2 - \sqrt{(\delta + \lambda\delta^{-1} - \nu^2)^2 - 4(\lambda - \frac{\delta\nu^2}{x})}}{2(\lambda - \frac{\delta\nu^2}{x})}$$

Under assumption (C.22), note that both roots are real. Furthermore, both roots are strictly positive and distinct for any $x > \delta\nu^2/\lambda$. In such case, we have $f(q,x) > 0$ for all $q \in (q_-(x), q_+(x))$. Thus, the inequality (C.26), combined with the fact that

$$\underline{q}_t^* \in \Big(q_-(\tilde{x}_\tau^*),\, q_+(\tilde{x}_\tau^*)\Big)$$

proves that $f(q_\tau^*; \tilde{x}_\tau^*) > 0$.

Finally, we may define a sequence of stopping times as follows. Let $\tau_0 := \tau$ and define recursively

$$\tau_{k+1} := \Big\{ t > \tau_k : q_t^* \leq \underline{q}_t^* := \frac{\delta + \lambda\delta^{-1} - \nu^2}{2(\lambda - \delta\nu^2/\tilde{x}_t^*)} \Big\}.$$

The same method above can used to prove that $dq_{\tau_k}^* > 0$ for any $k$, which implies $\tau_{k+1} > \tau_k$ almost-surely. Then, in each time interval $(\tau_k, \tau_{k+1})$, we have that $q_t^* \geq \underline{q}_t^*$. Furthermore, we have $\underline{q}_t^* > \frac{\delta + \lambda\delta^{-1} - \nu^2}{2\lambda} > 0$, following the proof method above. By piecing together the sequences of stopped processes, this completes the proof that $(q_t^*)_{t \geq 0} > 0$ almost-surely, as long as $\psi_{\tau_k}^* = 0$ for each $k$. □

## C.2 Debt overhang as a "stabilizing force"

In this section, we sketch an economy where firms face an investment problem, subject to neoclassical adjustment costs and debt-overhang. The result is a version of Q-theory, but with underinvestment. Because the predictions of this theory are so well-established, at some points we make reduced-form assumptions to simplify the analysis and illustrate our main points on stability.

**Firms.** There are a continuum of firms in each location $n$, each employing a linear technology with productivity $a$ and capital as the sole input. The evolution of firm-level capital is

$$dk_{n,t}^{(j)} = k_{n,t}^{(j)}[\iota_{n,t}^{(j)} - \kappa]dt + k_{n,t}^{(j)}\hat{\sigma}d\hat{B}_{n,t}^{(j)},$$

where $\iota$ is the endogenous investment rate, $\kappa$ is the exogenous depreciation rate, and $B^{(j)}$ is an idiosyncratic Brownian shock. The cost of making investment $\iota k$ is given by $\Phi(\iota)k$, where $\Phi(\cdot)$ is a convex adjustment cost function. Thus, the investment-production block has the standard homogeneity property in capital.

For this section only, we denote by $q_{n,t}^{(j)}$ the location-$n$ average value of capital to equity, i.e. "average Q" (this will not be the same as the price-dividend ratio that is called "$q$" in the main text, because the dividend is output minus investment). Thus, the value of firm $j$ is given by $q_{n,t}^{(j)}k_{n,t}^{(j)}$.

We also assume that all firms have long-term debt outstanding, in fact a perpetuity with a fixed and continuously-paid coupon as in Leland (1994) and its descendent papers, without microfounding the reasons for why (e.g., debt tax shield), as this is unimportant. Furthermore, to keep things simple, we assume existing firms can never issue new debt. Finally, firms default optimally, subject to some default costs that are proportional to the firm's capital (these can be redistributed to households to create no deadweight loss). Under these conditions, a typical finding is (see for example Hennessy, 2004, Proposition 2)

$$\tilde{q}_{n,t}^{(j)} := \text{marginal value of capital to equity} < \text{average value of capital to equity} = q_{n,t}^{(j)}.$$

Moreover, essentially by definition of $\tilde{q}$, the optimal investment satisfies $\tilde{q}_{n,t}^{(j)} = \Phi'(\iota_{n,t}^{(j)})$ (see for example Hennessy, 2004, equation 11). Thus, we see that $q_{n,t}^{(j)} > \Phi'(\iota_{n,t}^{(j)})$. The lack of equality here measures the deviation from neoclassical Q-theory.

Despite this deviation, we have the following property. Since $q_{n,t}^{(j)}$ increases with $\tilde{q}_{n,t}^{(j)} = \Phi'(\iota_{n,t}^{(j)})$, and since $\Phi$ is a convex function, we have $\iota_{n,t}^{(j)}$ increasing in $q_{n,t}^{(j)}$. We will furthermore make the reduced-form assumption that $\iota_{n,t}^{(j)} = \iota(q_{n,t}^{(j)})$ for some univariate increasing function $\iota(\cdot)$. This assumption is quite benign as it is typically satisfied in applications, because $\tilde{q}_{n,t}^{(j)}$, hence $q_{n,t}^{(j)}$, will typically be monotonic functions of the underlying firm-level state (e.g., leverage ratio).

In summary, we have the following two firm-level properties under debt overhang:

$$q_{n,t}^{(j)} > \Phi'(\iota_{n,t}^{(j)}) \tag{C.27}$$

$$\iota'(q_{n,t}^{(j)}) > 0. \tag{C.28}$$

Condition (C.27) captures the specific debt-overhang mechanism, whereas condition (C.28) is much more general and applies in almost any investment model. With a more general contractual structure, DeMarzo et al. (2012) also obtains these two results.

**Aggregation.** We will now make two assumptions that are mainly for tractability in aggregation. First, when a firm defaults and exits, it is replaced by another firm with the same identity $j$ that inherits the defaulting capital stock. We assume this new entrant issues new debt is such that the aggregate location-$n$ value of debt outstanding is always a constant fraction of total location-$n$ capital; i.e., total location-$n$ value of debt is always $\beta k_{n,t}$. Alternatively, this proportionality of aggregate debt to capital could be ensured by augmenting the model with a time-varying exogenous exit rate, but allowing new entrants to issue debt in an optimal way. Either way, this set of assumptions implies it suffices to study equity.

Second, we make assumptions to avoid studying the full cross-sectional distribution of firms within a location. We assume that properties (C.27)-(C.28) also hold in the aggregate at each location, and we will presume a certain approximate aggregation on investment and investment costs. In particular, let us define the appropriate aggregates, for capital, average Q, and investment:

$$k_{n,t} := \int k_{n,t}^{(j)} dj$$

$$q_{n,t} := \frac{1}{k_{n,t}} \int q_{n,t}^{(j)} k_{n,t}^{(j)} dj$$

$$\iota_{n,t} := \frac{1}{k_{n,t}} \int \iota(q_{n,t}^{(j)}) k_{n,t}^{(j)} dj.$$

As an approximation, we assume the existence of functions $(\bar{\iota}, \bar{\Phi})$ such that the following hold:

$$\bar{\iota}(q_{n,t}) \approx \int k_{n,t}^{(j)} \iota(q_{n,t}^{(j)}) dj \tag{C.29}$$

$$k_{n,t} \bar{\Phi}(\bar{\iota}(q_{n,t})) \approx \int k_{n,t}^{(j)} \bar{\Phi}(\bar{\iota}(q_{n,t}^{(j)})) dj. \tag{C.30}$$

The nature of these approximations is to say that aggregate location-$n$ investment is solely a function of aggregate average Q, rather than the full cross-sectional distribution of average Q's. Furthermore, we assume the following aggregate versions of properties (C.27)-(C.28), i.e.,

$$q_{n,t} > \bar{\Phi}'(\bar{\iota}_{n,t}) \tag{C.31}$$
$$\bar{\iota}'(q_{n,t}) > 0. \tag{C.32}$$

We conjecture these properties would go through in a full analysis of equilibrium using the cross-sectional distribution of firm size and Q, but this is beyond the scope of this paper. As we make these aggregation approximations, note that we also assume the functions $(\bar{\iota}, \bar{\Phi})$ are independent of location $n$.

**Stability.** Now, we can proceed to study stability. The aggregate portfolio of location-$n$ firms' liabilities (debt plus equity) has value $(\beta + q_{n,t})k_{n,t}$, which is a claim to the profits $\int (a - \Phi(\iota_{n,t}^{(j)}))k_{n,t}^{(j)} dj$. Based on approximation (C.30), this aggregate profit can be approximately written $(a - \bar{\Phi}(\bar{\iota}(q_{n,t})))k_{n,t}$. Furthermore, the return on this portfolio is deterministic, given that all fundamental shocks are idiosyncratic (hence defaults will be idiosyncratic), and thus the return must equal the riskless bond return $r_t$ in equilibrium. Thus, $q_{n,t}$ evolves deterministically, and the (approximate) valuation equation states

$$\frac{a - \bar{\Phi}(\bar{\iota}(q_{n,t}))}{q_{n,t}} + \bar{\iota}(q_{n,t}) - \kappa + \frac{\dot{q}_{n,t}}{q_{n,t}} = r_t. \tag{C.33}$$

**Lemma C.2.** *Suppose the number of locations $N$ is large enough, that approximations (C.29)-(C.30) hold, and that properties (C.31)-(C.32) hold with sufficient gaps between the left- and right-hand-sides (i.e., underinvestment is large enough). Then, the equilibrium of the model with debt overhang is locally-stable.*

***Proof of Lemma C.2.*** We start with approximate valuation equation (C.33). Differentiate $\dot{q}_{n,t}$ with respect to $q_{n,t}$ and $q_{-n,t}$ to obtain

$$\frac{d\dot{q}_{n,t}}{dq_{n,t}} = r_t + \kappa - \bar{\iota}(q_{n,t}) + \bar{\Phi}'(\bar{\iota}(q_{n,t}))\bar{\iota}'(q_{n,t}) - q_{n,t}\bar{\iota}'(q_{n,t}) + q_{n,t}\frac{dr_t}{dq_{n,t}}$$
$$\frac{d\dot{q}_{n,t}}{dq_{-n,t}} = q_{n,t}\frac{dr_t}{dq_{-n,t}}.$$

We will study these equations in the limit $N \to \infty$, which suffices, because the lemma allows us to later make $N$ large enough.

As $N \to \infty$, one can show that

$$r_t \to \delta - \kappa + \lim_{N\to\infty} \sum_{n=1}^{N} \frac{k_{n,t}}{\sum_{i=1}^{N} k_{i,t}} \bar{\iota}(q_{n,t}),$$

which has zero derivative with respect to $q_{i,t}$ for any $i$. Substituting this result for $r_t$, we obtain $d\dot{q}_{n,t}/dq_{-n,t} = 0$ and

$$\frac{d\dot{q}_{n,t}}{dq_{n,t}} = \delta + \underbrace{\lim_{N\to\infty} \sum_{m=1}^{N} \frac{k_{m,t}}{\sum_{i=1}^{N} k_{i,t}} \bar{\iota}(q_{m,t}) - \bar{\iota}(q_{n,t})}_{=0 \text{ in steady state}} - \left[q_{n,t} - \bar{\Phi}'(\bar{\iota}(q_{n,t}))\right]\bar{\iota}'(q_{n,t}).$$

The fact that the middle terms net out to zero in steady state is a consequence of the fact that $dk_{n,t} = k_{n,t}[\bar{\iota}(q_{n,t}) - \kappa]dt$, and all locations must experience the same growth rate $\bar{\iota}(q_{n,t}) - \kappa$ in steady state. Thus, we will have $d\dot{q}_{n,t}/dq_{n,t} < 0$, hence local stability by $d\dot{q}_{n,t}/dq_{-n,t} = 0$, if and only if

$$\left[q_{n,t} - \bar{\Phi}'(\bar{\iota}(q_{n,t}))\right]\bar{\iota}'(q_{n,t}) > \delta.$$

This will be true if properties (C.31)-(C.32) hold with sufficient gaps, as assumed. □

## C.3 Creative destruction as a "stabilizing force"

In this section, we consider another model that allows multiplicity. We show how an overlapping generations (OLG) "perpetual youth" economy – built upon Blanchard (1985) – augmented with a particular type of creative destruction – similar to Gârleanu and Panageas (2020) – creates a stabilizing force upon which extrinsic shocks can be layered. In particular, if new firm creation is more intense when asset valuations are low, the economy possesses a natural stabilizing force. A possible rationale for this feature is that when capital asset valuations are low, they make labor look relatively attractive, which offers a robust outside option for those new entrepreneurs willing to enter. The contribution relative to Gârleanu and Panageas (2020) is to show how this is possible with an arbitrary number of assets (corresponding to the $N$ locations) whose markets are, in addition, not integrated.

**Cohorts, Endowments, Markets.** In this model, all agents face a constant hazard rate of death $\beta > 0$, with all dying agents replaced by newborns (in the same location), so that population size is constant at 1. To keep matters simple, assume all locations have identical constant endowment growth rates and no shocks. That said, the endowment growth of an individual agent differs from the aggregate growth rate; this is the crucial ingredient in this model.

In particular, we assume some amount of *creative destruction*. The endowments of living agents decay at rate $\kappa_{n,t}$ (obsolescence rate), while newborn agents arrive to the economy with new trees of total size $\kappa_{n,t} + g$ (or, in per capita units, their individual trees are $(\kappa_{n,t} + g)/\beta$ in size). Specifically, the time-$t$ endowment accruing to location-$n$ agents born at time $s \leq t$ is

$$y_{n,t}^{(s)} = y_{n,t}(\kappa_{n,s} + g)\exp\Big[-\int_s^t(\kappa_{n,u} + g)du\Big].$$

Note that the aggregate endowment follows

$$dy_{n,t} = d\Big(\int_{-\infty}^t y_{n,t}^{(s)}ds\Big) = y_{n,t}^{(t)}dt + \int_{-\infty}^t dy_{n,t}^{(s)}ds = \underbrace{y_{n,t}(\kappa_{n,t} + g)dt}_{\text{newborn entry}} - \underbrace{y_{n,t}\kappa_{n,t}dt}_{\text{obsolescence}} = y_{n,t}g\,dt.$$

For now, we leave $\kappa_{n,t}$ unspecified, but note that its formulation will be the determinant of whether multiplicity is possible or not.

Agents can only trade in financial markets while alive. In addition to the tradability of claims to local endowments, agents can access a market for annuities that insures their death hazard and provides a stream of $\beta w_{n,t}^{(s)}$ of income per unit of time, where $w_{n,t}^{(s)}$ is the wealth of a location-$n$ agent born at time $s \leq t$. This assumption is standard in perpetual youth models.

**Solution.** Under these assumptions, one can show that agents consume $\delta + \beta$ fraction of their wealth, so that the bond market clearing condition (14) is replaced by

$$\sum_{n=1}^N \alpha_n q_{n,t} = (\delta + \beta)^{-1},$$

where $q_{n,t}$ is the (aggregated across cohorts) location-$n$ valuation ratio. Let $\xi_{n,t}$ denote the location-$n$ state-price density, which follows

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \pi_{n,t} dZ_{n,t}\Big].$$

We will continue to examine a bubble-free equilibrium, so that

$$q_{n,t} = \mathbb{E}_t\Big[\int_t^\infty \frac{\xi_{n,\tau}}{\xi_{n,t}}\frac{y_{n,\tau}^{(s)}}{y_{n,t}^{(s)}}d\tau\Big] \quad \text{(for any birth-date } s \leq t\text{, this yields the same answer).}$$

Critically, this valuation does not incorporate wealth gains due to entry of future newborns (i.e., this is the value of alive firms). The dynamic counterpart of this valuation equation is, for some diffusion coefficient $\sigma_{n,t}^q$,

$$\frac{dq_{n,t}}{q_{n,t}} = \Big[r_t + \kappa_{n,t} - \frac{1}{q_{n,t}} + \sigma_{n,t}^q \pi_{n,t}\Big]dt + \sigma_{n,t}^q dZ_{n,t}. \tag{C.34}$$

The equilibrium riskless rate is obtained as follows. The goods market is integrated across locations, so the market clearing condition is given by

$$Y_t = \sum_{n=1}^N y_{n,t} = \sum_{n=1}^N \int_{-\infty}^t \beta e^{-\beta(t-s)} c_{n,t}^{(s)} ds.$$

Optimal consumption dynamics for alive agents are

$$\frac{dc_{n,t}^{(s)}}{c_{n,t}^{(s)}} = \Big[r_t - \delta + \pi_{n,t}^2\Big]dt + \pi_{n,t} dZ_{n,t},$$

whereas newborn agents consume

$$\beta c_{n,t}^{(t)} = \underbrace{(\delta+\beta)}_{\substack{\text{cons-wealth}\\ \text{ratio}}} \times \underbrace{(\kappa_{n,t}+g) y_{n,t} q_{n,t}}_{\text{newborn wealth}}.$$

Applying Itô's formula to goods market clearing, and using these results, we obtain

$$r_t = \delta + \beta - \sum_{n=1}^N x_{n,t}\pi_{n,t}^2 - (\delta+\beta)\sum_{n=1}^N \alpha_n q_{n,t}\kappa_{n,t}. \tag{C.35}$$

**Stability.** To see how the stabilizing force works, it is instructive to once again study the deterministic equilibrium in which extrinsic shocks have no volatility. Substituting (C.35) into (C.34) with $\sigma_{n,t}^q = 0$, we obtain

$$\dot{q}_{n,t} = \underbrace{-1 + (\delta+\beta)q_{n,t}}_{\text{unstable component}} - \underbrace{\Big[(\delta+\beta)\sum_{i=1}^N \alpha_i q_{i,t}\kappa_{i,t} - \kappa_{n,t}\Big]q_{n,t}}_{\text{stabilizing force}} \quad \text{when} \quad \sigma_{i,t}^q = 0 \quad \forall i. \tag{C.36}$$

The first piece is the unstable component, propelling valuations further and further away from the "steady state" value $(\delta+\beta)^{-1}$. The second piece—capturing the relative amount of creative destruction in location $n$—is the stabilizing force.

Based on equation (C.36), we claim that if $\kappa_{n,t}$ decreases sufficiently rapidly as $q_{n,t}$ increases, then valuation dynamics are stable. Let $\kappa_{n,t} = \kappa(q_{n,t})$ for a decreasing function $\kappa(\cdot)$. Denote the steady-state mean and sensitivity of this function by $\bar{\kappa} := \kappa((\delta+\beta)^{-1})$ and $\lambda := -\kappa'((\delta+\beta)^{-1})$, respectively. Then, compute

$$\frac{\partial \dot{q}_n}{\partial q_m}\Big|_{q_i=(\delta+\beta)^{-1}\,\forall\, i} = \begin{cases} \delta+\beta-\lambda(\delta+\beta)^{-1}(1-\alpha_n)-\alpha_n\bar{\kappa}, & \text{if } m=n; \\ \lambda(\delta+\beta)^{-1}\alpha_m - \alpha_m\bar{\kappa}, & \text{if } m \neq n. \end{cases}$$

Construct the steady-state Jacobian matrix as

$$J := \left[\frac{\partial \dot{q}_n}{\partial q_m}\Big|_{q_i=(\delta+\beta)^{-1}\,\forall\, i}\right]_{1\le n,m\le N}. \tag{C.37}$$

Local stability of the steady-state can be determined by the eigenvalues of $J$. By the Gershgorin circle theorem, all of these eigenvalues will have strictly negative real parts if $J$ has negative diagonal elements and is diagonally dominant. This is easily guaranteed by making $\bar{\kappa}$ and $\lambda$ large enough, meaning the amount of creative destruction and its sensitivity to prices are both large enough. The result is summarized in the following lemma, with the proof omitted.

**Lemma C.3.** *Assume $\bar{\kappa} > \delta+\beta$ and $\lambda > (\delta+\beta)\bar{\kappa}$. Then, all eigenvalues of $J$ have strictly negative real parts. Consequently, the equilibrium of the creative destruction model is locally stable.*

# D International model of Section 4.3

**Derivation of equilibrium.** As before, let $Y_t := \sum_{n=1}^N y_{n,t}$ be aggregate tradable consumption, and define (tradable) consumption shares $x_{n,t} := c_{n,t}/Y_t$ and (tradable) endowment shares $\alpha_{n,t} := y_{n,t}/Y_t$. Country $n$ state price density $\xi_{n,t}$ still evolves according to Eq. (A.1), repeated here for convenience

$$\frac{d\xi_{n,t}}{\xi_{n,t}} = -r_t dt - \eta_t dB_t - \hat{\eta}_t \cdot d\hat{B}_t - \pi_{n,t} dZ_{n,t}.$$

The representative agent of country $n$ maximizes (29) subject to the lifetime budget constraint

$$w_{n,0} = \mathbb{E}_0\Big[\int_0^\infty \frac{\xi_{n,t}}{\xi_{n,0}}(c_{n,t}+p_{n,t}\hat{c}_{n,t})dt\Big]. \tag{D.1}$$

Solving this maximization problem delivers FOCs $e^{-\delta t}\phi c_{n,t}^{-1} = \xi_{n,t}$ and $e^{-\delta t}(1-\phi)\hat{c}_{n,t}^{-1} = \xi_{n,t}p_{n,t}$, which together imply the expenditure shares in (31). To obtain the dynamic consumption rule, substitute these FOCs back into the budget constraint (D.1) to get $c_{n,0} + p_{n,0}\hat{c}_{n,0} = \delta w_{n,0}$. This is equivalent to Eq. (30) after using the definition of the price and quantity index $P_{n,t}C_{n,t} = c_{n,t} + p_{n,t}\hat{c}_{n,t}$. Then, the optimal dynamics of non-tradable consumption $c_{n,t}$, expenditure $P_{n,t}C_{n,t}$, and wealth $w_{n,t}$ all take the same form, namely

$$\frac{dc_{n,t}}{c_{n,t}} = \frac{dw_{n,t}}{w_{n,t}} = \Big[r_t - \delta + \eta_t^2 + \|\hat{\eta}_t\|^2 + \pi_{n,t}^2\Big]dt + \eta_t dB_t + \hat{\eta}_t \cdot d\hat{B}_t + \pi_{n,t}dZ_{n,t}. \tag{D.2}$$

As in the baseline model, using $\sum_{n=1}^N dc_{n,t} = dY_t$ and matching drifts and diffusions, we obtain the interest rate in Eq. (A.14) and risk prices in Eqs. (A.15)-(A.17), all repeated here for convenience

$$r_t = \delta + g_t - \nu^2 - \sum_{n=1}^N x_{n,t}\pi_{n,t}^2, \quad \text{and} \quad \eta_t = \nu, \quad \text{and} \quad \hat{\eta}_t = 0, \quad \text{and} \quad \sum_{n=1}^N x_{n,t}\pi_{n,t}M_n = 0.$$

Therefore, the dynamics of $x_{n,t}$ are identical to the baseline model Eq. (A.18). Finally, use the tradable expenditure share rule to write aggregate wealth as

$$\sum_{n=1}^{N} w_{n,t} = \sum_{n=1}^{N} \frac{c_{n,t}}{\phi\delta} = \frac{Y_t}{\phi\delta}. \tag{D.3}$$

So far, this is nearly identical to the baseline model. The step that diverges from the baseline model, which we tackle next, regards the local equity pricing equation.

The return on local equity $dR_{n,t}$ is defined by

$$dR_{n,t} := \frac{1}{q_{n,t}}dt + \frac{d(q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t}))}{q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})},$$

where the valuation ratio $q_{n,t}$ has dynamics of the form

$$\frac{dq_{n,t}}{q_{n,t}} = \mu^q_{n,t}dt + \varsigma^q_{n,t}dB_t + \hat{\varsigma}^q_{n,t} \cdot d\hat{B}_t + \sigma^q_{n,t}dZ_{n,t}.$$

Apply Itô's formula to $d(q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t}))$, using the fact from Eqs. (30)-(31) that $p_{n,t}\hat{y}_{n,t} = \frac{1-\phi}{\phi}c_{n,t}$ and also using the equilibrium risk prices and interest rate, to obtain

$$\begin{aligned} dR_{n,t} = {} & \frac{1}{q_{n,t}}dt + \mu^q_{n,t}dt + \varsigma^q_{n,t}dB_t + \hat{\varsigma}^q_{n,t} \cdot d\hat{B}_t + \sigma^q_{n,t}dZ_{n,t} \\ & + \frac{\phi y_{n,t}}{\phi y_{n,t} + (1-\phi)c_{n,t}}\Big([g_{n,t} + \nu\varsigma^q_{n,t} + \hat{\nu}_{n,t} \cdot \hat{\varsigma}^q_{n,t}]dt + \nu dB_t + \hat{\nu}_{n,t} \cdot d\hat{B}_t\Big) \\ & + \frac{(1-\phi)c_{n,t}}{\phi y_{n,t} + (1-\phi)c_{n,t}}\Big([r_t - \delta + \nu^2 + \pi^2_{n,t} + \nu\varsigma^q_{n,t} + \pi_{n,t}\sigma^q_{n,t}]dt + \nu dB_t + \pi_{n,t}dZ_{n,t}\Big) \end{aligned} \tag{D.4}$$

Consequently, the no-arbitrage pricing equation is (after substituting the equilibrium risk prices and doing extensive algebra)

$$\mu^q_{n,t} = \delta - \frac{1}{q_{n,t}} + \frac{\phi y_{n,t}}{\phi y_{n,t} + (1-\phi)c_{n,t}}\Big(r_t - \delta + \nu^2 + \pi_{n,t}\sigma^q_{n,t} - g_{n,t} - \hat{\nu}_{n,t} \cdot \hat{\varsigma}^q_{n,t}\Big). \tag{D.5}$$

Eq. (D.5) characterizes the critical dynamical system of the model.

To connect the risk prices to the valuation dynamics, recall the dynamic budget constraint

$$dw_{n,t} = (w_{n,t}r_t - P_{n,t}C_{n,t})dt + \vartheta_{n,t}(\eta_t dt + dB_t) + \hat{\vartheta}_{n,t} \cdot (\hat{\eta}_t dt + d\hat{B}_t) + \theta_{n,t}(dR_{n,t} - r_t dt). \tag{D.6}$$

First, using local equity market clearing $\theta_{n,t} = q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})$ and matching the $dZ_{n,t}$ loadings in Eq. (D.6) to those in Eq. (D.2), we have

$$\pi_{n,t} = \frac{q_{n,t}(y_{n,t} + p_{n,t}\hat{y}_{n,t})}{w_{n,t}}\Big(\sigma^q_{n,t} + \frac{p_{n,t}\hat{y}_{n,t}}{y_{n,t} + p_{n,t}\hat{y}_{n,t}}\pi_{n,t}\Big)$$

Using $p_{n,t}\hat{y}_{n,t} = (1-\phi)\delta w_{n,t}$ and $c_{n,t} = \phi\delta w_{n,t}$, and then solving this equation for $\pi_{n,t}$, we obtain

$$\pi_{n,t} = \delta\frac{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}{x_{n,t}(1-(1-\phi)\delta q_{n,t})}q_{n,t}\sigma^q_{n,t}. \tag{D.7}$$

Using Eq. (D.7) inside $\sum_{n=1}^{N} x_{n,t}\pi_{n,t}M_n = 0$, we have the following restriction on sunspot volatilities:

$$0 = \sum_{n=1}^{N} \frac{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}{1-(1-\phi)\delta q_{n,t}} q_{n,t}\sigma^q_{n,t} M_n. \tag{D.8}$$

Second, summing both Eqs. (D.6) and (D.2) over $n$, using $\eta_t = \nu$ and $\hat{\eta}_t = 0$, using futures market clearing conditions $\sum_{n=1}^{N} \vartheta_{n,t} = 0$ and $\sum_{n=1}^{N} \hat{\vartheta}_{n,t} = 0$, and using the aggregate wealth constraint (D.3), we obtain

$$\nu = \delta \sum_{n=1}^{N} q_{n,t}(\phi\alpha_{n,t} + (1-\phi)x_{n,t})(\nu + \varsigma^q_{n,t})$$

$$0 = \sum_{n=1}^{N} q_{n,t}(\phi\alpha_{n,t} + (1-\phi)x_{n,t})(\hat{\nu}_{n,t} + \hat{\varsigma}^q_{n,t})$$

This completes the set of equilibrium equations, analogously to Appendix A.

**Construction and stability of sunspot equilibria.** Let $M$ be an $N \times N$ matrix with $\mathrm{rank}(M) < N$, let $v^* := (v_1^*, \ldots, v_N^*)'$ be in the null-space of $M'$, and let $\psi_t$ be a positive scalar process. Since $0 = \sum_{n=1}^{N} x_{n,t}\pi_{n,t}M_n$ holds, as in the baseline model, we thus construct a candidate equilibrium with

$$\pi_{n,t} = \frac{\delta\psi_t}{x_{n,t}} v_n^*. \tag{D.9}$$

By Eq. (D.7), we then have

$$\frac{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}{1-(1-\phi)\delta q_{n,t}} q_{n,t}\sigma^q_{n,t} = \psi_t v_n^*, \tag{D.10}$$

From this point, the arguments in Theorem 1 go through without modification, so we will have an equilibrium if $(q_{n,t})_{n=1}^{N}$ are positive, bounded processes and if $\lim_{T\to\infty} \mathbb{E}_t[e^{-\delta T} x_{n,T}^{-1}] = 0$ for each $n$. Given the growth-valuation functional form $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$, these conditions can be verified in the same was as in the proof of Proposition 1. We only sketch the intuition, given the similarity to Proposition 1.

In particular, to see that the valuation dynamics are stable, substitute $\hat{\varsigma}^q_{n,t} = 0$, $r_t = \delta + g - \nu^2 - \sum_{n=1}^{N} x_{n,t}\pi^2_{n,t}$, $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$, and Eqs. (D.9)-(D.10) into Eq. (D.5) to get

$$\begin{aligned} q_{n,t}\mu^q_{n,t} = &-1 + \Big(\delta + \lambda\delta^{-1}\frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}\Big) q_{n,t} - \frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}\lambda q^2_{n,t} \\ &+ \Big(\frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}\Big)^2 \frac{\delta(\psi_t v_n^*)^2}{\phi\alpha_{n,t}x_{n,t}}[1-(1-\phi)\delta q_{n,t}] - \frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}\Big(\delta^2\psi_t^2 \sum_{i=1}^{N}\frac{(v_i^*)^2}{x_{i,t}}\Big) q_{n,t}. \end{aligned} \tag{D.11}$$

When $\psi_t = 0$, the entire second line of Eq. (D.11) vanishes. In that case, we can see that $q_{n,t}\mu^q_{n,t}$ is decreasing with respect to $q_{n,t}$ if and only if

$$2\frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}\lambda q_{n,t} > \delta + \lambda\delta^{-1}\frac{\phi\alpha_{n,t}}{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}$$

For $q_{n,t} = \delta^{-1}$ (steady state), this condition becomes

$$\lambda > \delta^2 \frac{\phi\alpha_{n,t} + (1-\phi)x_{n,t}}{\phi\alpha_{n,t}}$$

Therefore, if $\lambda$ is large enough (e.g., larger than $K\delta^2$), then we can construct a sunspot equilibrium in which volatility $\psi_t$ vanishes whenever either (i) $q_{n,t}$ deviates too far from "steady state"; or (ii) $\frac{\phi\alpha_{n,t}+(1-\phi)x_{n,t}}{\phi\alpha_{n,t}}$ becomes too large (e.g., it reaches $K$ in the example where $\lambda > K\delta^2$).