Fiscal Policies as Equilibrium Selection in Uncertain Environments*

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Abstract

We provide an extended analysis of fiscal policies that eliminate "uncertainty trap" equilibria in standard New Keynesian models (Khorrami and Mendo, 2025). A large class of fiscal policies eliminates uncertainty traps and ensures determinacy. This uniqueness holds in many contexts, including: under an interest rate peg or Taylor rule, with short-term or long-term debt, and with fiscal rules. In addition to always-active fiscal regimes, other valid policies include fiscal backstops promising occasional interventions, potentially only in extreme recessions.

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This paper revisits the uncertainty trap equilibria unveiled by Khorrami and Mendo (2025) and Lee and Dordal i Carreras (2024). These uncertainty traps emerge when solving the New Keynesian (NK) model without linearization, and remain equilibria even when monetary policy adheres to the Taylor principle. Of course, the uncertainty trap problem is especially acute when monetary policy is constrained, say by a zero lower bound. In this sense, the uncertainty trap phenomenon is pervasive to NK models. At the same time, Khorrami and Mendo (2025) show that uncertainty traps are reasonable. First of all, they rely on a very natural economic intuition whereby higher uncertainty induces precautionary savings hence a demand recession. Second, they induce countercyclical volatility, as is observed in the data, of empirically reasonable magnitude. And third, uncertainty traps do not rely on self-fulfilling (hyper-)inflations or deflations, and so are distinct from some existing multiplicity concerns in NK models (e.g., Cochrane, 2011; Benhabib et al., 2001).

As demonstrated in Khorrami and Mendo (2025), some types of fiscal policies can completely eliminate uncertainty traps, regardless of the monetary policy stance, thereby solving the equilibrium selection conundrum. In particular, a class of "active fiscal policies" (in the language of Leeper, 1991) with exogenous surplus-to-GDP ratios can eliminate all self-fulfilling uncertainty. This paper extends the equilibrium selection result by generalizing to much richer set of fiscal policies.

To understand the result, note that active fiscal policy, in contrast to passive policy, does not pledge to ensure debt sustainability under any and every possible shock. Instead, some shocks can be "unbacked" and must be *absorbed* via inflation, debt valuations, or economic growth.

If absorption cannot happen, then the uncertainty trap equilibrium is ruled out. Sticky prices say that prices cannot jump arbitrarily, and so the price level cannot absorb such a shock. What about the nominal debt value? In the baseline with short-term debt, the debt price is 1, while the quantity of debt is simply determined by the flow government budget constraint; thus, the nominal value of debt is pinned down and cannot absorb the shock either. In the extension with long-term debt, the bond price is an additional forward-looking variable that could potentially respond to shocks. But with the additional variable comes an additional constraint, the bond-pricing equation, which is inconsistent with the originally conjectured output shock. Given absorption fails, active fiscal policy effectively steers aggregate demand, disciplining its response to aggregate shocks and closing the door to the volatility required to sustain uncertainty traps. The novelty and difficulty of this analysis stems from the nonlinearity of the environment.

In the last part of the paper, we allow fiscal policy to switch over time between "passive" and "active" regimes. We start by providing a general characterization of when such a policy can select equilibria and when it cannot. We then consider a *fiscal backstop*, in which an aggressive fiscal regime emerges endogenously in extreme recessions. Think of this as a fiscal policy that abandons its debt sustainability focus in recessions. And agents understand ex-ante this recessionary switch. We demonstrate that such a policy profile can indeed eliminate uncertainty traps, so long as fiscal policy is active often enough and acts aggressive-enough when it is active. Our results on equilibrium selection with this type of regime-switching policy are also novel.

A literature on the fiscal theory of the price level (FTPL) also explores determinacy in NK models under various policy profiles.[1] Unlike most of this literature, we analyze the fully nonlinear, stochastic, global dynamics of the model. Some papers have extended FTPL to stochastic nonlinear stochastic environments, but almost exclusively with *flexible prices* (Bassetto and Cui, 2018; Brunnermeier et al., 2023, 2024).[2] Two exceptions that do allow sticky prices, but sidestep our determinacy questions, are Mehrotra and Sergeyev (2021) and Li and Merkel (2025). Mehrotra and Sergeyev (2021) study fiscal sustainability with real debt, but in a setting with exogenous output. Li and Merkel (2025) study FTPL in a NK model with idiosyncratic risk, which can induce a government debt bubble; however, they avoid determinacy questions by assuming that endogenous objects like inflation and the output gap are Markovian in exogenous states and government bonds outstanding (i.e., they assume an MSV solution). Overall, in conjunction with Khorrami and Mendo (2025), we provide the first formal nonlinear FTPL-style analysis in textbook sticky-price models. In the present paper, we do so across many environments including surplus shocks, long-term debt, surplus rules, and active/passive regimes.

1 Equilibrium Conditions in a Nonlinear NK Model

We briefly summarize the equilibrium with uncertainty traps in Khorrami and Mendo (2025). The model is a canonical New Keynesian (NK) economy with complete markets and nominal rigidities. We study the model in continuous time for tractability. There is no fundamental uncertainty in preferences or technologies. To permit coordination equilibria, we introduce a sunspot shock: a one-dimensional Brownian motion Z. In addition, we allow a set of fiscal shocks that are adapted to the independent vector

[1] Seminal contributions to the FTPL include Leeper (1991), Sims (1994), Woodford (1994), Woodford (1995), Kocherlakota and Phelan (1999), and Cochrane (2001). Cochrane (2023) synthesizes many results.

[2] Others studying the FTPL in nonlinear, but deterministic, environments with liquidity premia include Berentsen and Waller (2018), Williamson (2018), Andolfatto and Martin (2018), and Miao and Su (2024).

Brownian motion $\mathcal{Z}$. All random processes will be adapted to Z and $\mathcal{Z}$.

The representative agent has rational expectations and time-separable utility with discount rate ρ, risk aversion γ, and labor disutility parameter φ:

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(\log(C_t) - \frac{L_t^{1+\varphi}}{1+\varphi}\Big)dt\Big]. \tag{1}$$

Consumption C_t has the nominal price P_t and labor L_t earns the nominal wage W_t.

The consumption good is produced by a linear technology $Y_t = L_t$. Behind the aggregate production function is a structure common to most of the NK literature. In particular, there are a continuum of firms who produce intermediate goods using labor in a linear technology. These intermediate goods are aggregated by a competitive final goods sector. The elasticity of substitution across intermediate goods is a constant ε. The intermediate-goods firms behave monopolistically competitively and set prices strategically, subject to quadratic price adjustment costs a la Rotemberg (1982).

Finally, we assume monetary policy follows a standard Taylor rule with feedback from output and inflation into interest rates.

In this environment, the following three equations characterize equilibrium:

$$dx_t = \Big[\iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2 + \frac{1}{2}|\varsigma_{x,t}|^2\Big]dt + \sigma_{x,t}dZ_t + \varsigma_{x,t}d\mathcal{Z}_t \tag{2}$$

$$d\pi_t = \Big[\rho\pi_t - \kappa f(x_t)\Big]dt + \sigma_{\pi,t}dZ_t + \varsigma_{\pi,t}d\mathcal{Z}_t. \tag{3}$$

$$\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) \tag{4}$$

Equation (2) is the nonlinear IS curve. The key novelty is σ_x^2 and $|\varsigma_x|^2$, which encode precautionary savings. Equation (3) is the nonlinear Phillips curve characterizing inflation $\pi_t := \dot{P}_t/P_t$, with parameter κ denoting a composite price-stickiness parameter (as $\kappa \to 0$, prices become infinitely rigid), and with $f(x) := \frac{e^{(1+\varphi)x}-1}{1+\varphi}$. Because it is less crucial to our analysis, we sometimes linearize this Phillips curve, i.e., we replace $f(x)$ with its linear approximation x. Equation (4) is the Taylor rule of monetary policy, with a potentially nonlinear reaction function Φ. The conventional linear case is $\Phi(x,\pi) = \phi_x x + \phi_\pi \pi$.

An equilibrium is a stochastic process for (x,π,ι) that satisfies (2)-(4) subject to a non-explosiveness condition that ensures all transversality conditions hold:

Condition 1. *A non-explosive allocation has* $\mathbb{P}\{|x_t| < \infty,\ |\pi_t| < \infty;\ \forall t \geq 0\} = 1$,

$$\limsup_{T\to\infty} \mathbb{E}|x_T| < \infty, \quad \limsup_{T\to\infty} \mathbb{E}|\pi_T| < \infty, \quad \text{and} \quad \limsup_{T\to\infty} \mathbb{E}[e^{(1+\varphi)x_T - \rho T}] = 0. \tag{5}$$

This non-explosiveness condition is a bit stronger than necessary to ensure transversality, but it still permits multiplicity of equilibria. Indeed, Khorrami and Mendo (2025) show that a large class of uncertainty traps satisfy (2)-(4) and Condition 1. Furthermore, given any Taylor rule, one can find an uncertainty trap equilibrium.

2 Fiscal Policy

We introduce fiscal policy, formulated via lump-sum taxation and government transfers to the representative household. The real primary surplus of the government is the difference between these taxes and transfers, which we denote by S_t. Fiscal policy is characterized by a specification for S_t, to be described shortly.

Surpluses may be non-zero, so the government borrows by issuing nominally riskless bonds in quantity B_t. To keep things tractable, let us assume that debt is coupon-free and has a constant exponential maturity structure. Per unit of time dt, a constant fraction βdt of outstanding debts mature, and their principal must be repaid. Denote the per-unit price of this debt by Q_t. The flow budget constraint of the government is thus

$$Q_t \dot{B}_t = \beta B_t - \beta B_t Q_t - P_t S_t. \tag{6}$$

This says that new net debt sales $\dot{B}_t + \beta B_t$, which garner price Q_t, plus primary surpluses $P_t S_t$ must be sufficient to pay back maturing debts βB_t. Taking $\beta \to \infty$ results in the standard benchmark with instantaneously-maturing debt, in which case $Q_t \to 1$ and the flow budget constraint would become $\dot{B}_t = \iota_t B_t - P_t S_t$. The nominal interest rate ι_t is still controlled by monetary policy.

With non-zero debt, equilibria must obey the household transversality condition for government debt holdings,

$$0 = \lim_{T \to \infty} \mathbb{E}_0 \Big[M_T \frac{Q_T B_T}{P_T} \Big], \tag{7}$$

where M is the real stochastic discount factor process. In this section, the transversality condition will be used extensively as a condition to trim equilibria. As is well known, the transversality condition implies the present-value formula for government debt:

$$\frac{Q_t B_t}{P_t} = \mathbb{E}_t \Big[\int_t^\infty \frac{M_u}{M_t} S_u du \Big]. \tag{8}$$

The real value of debt must equal the real present value of surpluses. In some arguments, it will be easier to use (8) rather than (7). We may rewrite the transversality and debt valuation equation in terms of the surplus-to-GDP ratio and the real debt-to-GDP ratio

$$s_t := \frac{S_t}{Y_t} \quad \text{and} \quad b_t := \frac{Q_t B_t}{P_t Y_t}. \tag{9}$$

Using these definitions, and using the consumption FOC $M_t = e^{-\rho t} Y_t^{-1}$, equations (7) and (8) can be written

$$0 = \lim_{T \to \infty} \mathbb{E}_0 \left[e^{-\rho T} b_T \right], \tag{10}$$

$$b_t = \mathbb{E}_t \Big[\int_t^\infty e^{-\rho(u-t)} s_u du \Big] \tag{11}$$

Since we plan to specify surpluses in terms of their ratio to GDP, these scaled versions of the conditions will be easier to use.

In addition to the conditions above, standard no-arbitrage asset-pricing implies that the per-unit bond price is given by

$$Q_t = \mathbb{E}_t \Big[\int_t^\infty \frac{M_T}{M_t} \frac{P_t}{P_T} \beta e^{-\beta(T-t)} dT \Big]. \tag{12}$$

In the above, debt is nominal, so it is priced using the nominal SDF M/P (intuitively, dividing by P converts a nominal cash flow into a real cash flow). This per-unit pricing equation is a restriction on the dynamics that Q_t can take.

Finally, we specify the dynamics of surpluses. We assume that surpluses feature a component that is exogenous, a component involving economic feedbacks from output and inflation, and a component including a feedback from outstanding government debt. We specify the surplus-to-GDP ratio $s_t := S_t / Y_t$ as

$$s_t = \hat{s}(\Omega_t) + \gamma(x_t, \pi_t) + \alpha_t b_t, \tag{13}$$

where $\hat{s}(\cdot)$ is a bounded continuous function of exogenous states Ω_t, where $\gamma(\cdot)$ is a bounded continuous function of the output gap and inflation, and where α_t is a potentially time- and state-dependent policy parameter, to be specified in examples below. The exogenous state variables Ω_t are driven by a multivariate Brownian motion $\mathcal{Z}$ that is independent of the shock Z:

$$d\Omega_t = \mu_\Omega(\Omega_t) dt + \varsigma_\Omega(\Omega_t) \cdot d\mathcal{Z}_t.$$

Together with a nominal interest rate rule for ι_t and a surplus rule for s_t, equilibrium is fully characterized by the dynamical system (2)-(3), combined with government transversality (7), or equivalently the valuation equation (8), as well as the debt pricing equation (12). We continue to require the non-explosion Condition 1.

Khorrami and Mendo (2025) analyzed fiscal policies with $\beta = \infty$ (short-term debt), with $\gamma = 0$ (no feedback from x and π), and with α_t time-invariant. In that case, if $\alpha > 0$, fiscal policy is "passive" and the uncertainty trap equilibria remain. If $\alpha = 0$, fiscal policy is "active" and eliminates uncertainty trap equilibria. We also use this active/passive language here (following Leeper, 1991). For concreteness, we restate two results of Khorrami and Mendo (2025) about whether fiscal policy selects an equilibrium.

Theorem 1 (Uncertainty trap with always-passive fiscal)**.** *Suppose $\beta = \infty$, $\gamma(\cdot) = 0$, and $\alpha_t = \bar{\alpha} > 0$. Suppose monetary policy is active in the sense that $\Phi(x, \pi)$ responds sufficiently to x and π. Then, there exist a class of stationary sunspot equilibria with $\sigma_x \neq 0$.*

Theorem 2 (Unique equilibrium with always-active fiscal)**.** *Suppose $\beta = \infty$, $\gamma(\cdot) = 0$, and $\alpha_t = 0$. Then, any equilibrium must have $\sigma_x = 0$, hence uncertainty traps cannot exist.*

We aim to generalize these arguments. The plan is as follows. First, given our α_t can be time-varying, we will generalize the logic of how a non-trivial debt valuation equation depends on the sign of α_t. The results are given as general tools in Section 2.1. Second, we use the debt valuation equation to generalize our equilibrium selection arguments. To this end, we will analyze state-dependent fiscal activism (i.e., time-varying α), long-term debt ($\beta < \infty$), and feedback rules ($\gamma \neq 0$). For analytical tractability, however, we analyze these extensions one at a time: Theorem 3 covers state-dependent fiscal activism with short-term debt and without fiscal rules; Theorem 4 covers long-term debt with constant fiscal activism and without fiscal rules; and Theorem 5 covers fiscal rules with constant fiscal activism and short-term debt.

One technical restriction is required to prove these next results. In particular, we restrict attention to a class of "Markovian equilibria" in which everything is a function of (x, Ω). This restriction still permits a somewhat general analysis that nests all the self-fulfilling equilibria developed in Khorrami and Mendo (2025).

Definition 1. An (x, Ω)-*Markov equilibrium* is a non-explosive equilibrium such that the debt feedback α_t, inflation π_t, and volatilities $\sigma_{x,t}, \varsigma_{x,t}$ are functions of (x_t, Ω_t).

2.1 General classification results: what is passive and what is active?

We start by providing two novel general tools to diagnose whether a complex policy profile is ultimately passive or active. Our first such tool characterizes passive fiscal policies, in the sense that the government remains solvent irrespective of the economic dynamics.

Lemma 1 (Passive fiscal)**.** *Suppose* $\alpha_t \geq 0$ *and*

$$\mathbb{P}\Big\{ \int_t^\infty \alpha_u du = +\infty, \quad \forall t \geq 0 \Big\} = 1. \tag{14}$$

Then, transversality condition (7) holds, irrespective of the path of (x_t, π_t)*.*

Lemma 1 is used to "rule in" equilibria. If $\alpha_t \geq 0$ and condition (14) holds within a conjectured equilibrium, then the fiscal policy profile does nothing to rule out the conjecture, precisely because the debt valuation equation is redundant to the other equations. Clearly, Lemma 1 covers the conventional example policy with $\alpha_t = \bar{\alpha} > 0$. But it can also cover more complex policies with time-varying α_t. Some examples follow at the end of this section.

Whereas Lemma 1 is a tool to "rule in" equilibria, we also provide a converse result that will be sufficient to "rule out" equilibria. This is a characterization of what active fiscal policy looks like, in the sense that an additional substantive condition arises—a debt valuation-like equation—that ultimately constrains the economic dynamics.

Lemma 2 (Active fiscal)**.** *Suppose* $\sup_t \alpha_t < \rho$ *and*

$$\mathbb{P}\Big\{ \int_t^\infty \alpha_u du < +\infty, \quad \forall t \geq 0 \Big\} = 1. \tag{15}$$

Then, transversality condition (7) implies debt-to-GDP is given by

$$b_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} e^{\int_t^u \alpha_z dz} (\hat{s}_u + \gamma_u) du \Big], \tag{16}$$

where $\hat{s}_t = \hat{s}(\Omega_t)$ *and* $\gamma_t = \gamma(x_t, \pi_t)$*.*

The critical valuation-like equation (16) looks much like the generic debt valuation equation (8), but it is not exactly the same for two reasons. First, it is not always true that the generic valuation equation (8), which holds in every equilibrium, coincides with the valuation-like equation (16). These two conditions are only equivalent under the assumptions of Lemma 2. Second, the valuation-like equation has purged the surpluses of their endogenous response to debt, making it easier to analyze.

We now consider several examples to illustrate the power of Lemmas 1-2. The first example is very simple and involves time-dependent regime-switching.

Example 1. *Let $\bar{\alpha} \in (0, \rho)$. Consider the two fiscal policies α_t^A and α_t^B, defined by*

Dates	α_t^A	α_t^B
$t < T$	0	$\bar{\alpha}$
$t \geq T$	$\bar{\alpha}$	0

Policy A is the type of policy discussed in Angeletos et al. (2024). It is clear that, regardless of the value of T, $\int_0^\infty \alpha_t^A dt = +\infty$, and so Lemma 1 applies. Therefore, this is passive fiscal, even though government here refuses to accommodate its debt increases for an arbitrarily long time. Angeletos et al. (2024) use this to argue that FTPL equilibrium selection is not robust to very far in the future revisions in the policy regime.

But one can easily make the exact opposite argument about policy B. Regardless of how large T is, $\int_0^\infty \alpha_t^B dt < +\infty$, and so Lemma 2 applies. This policy is active. In that sense, the FTPL is indeed robust to arbitrarily long deviations from active fiscal policy.

The problem here is the strangeness of the limit $T \to \infty$ for when a permanent regime-switch occurs. No matter how large T is, there is infinite time afterward. That is why the two examples A and B are not helpful to determine whether or not FTPL's conclusions are fragile or not.

As the next example illustrates, our lemmas can easily address even more complex policy profiles that involve state-dependent regime-switching. This example fiscal policy will be unsuccessful in providing equilibrium selection, despite appearing to have a substantial degree of "fiscal activism."

Example 2 (Recessionary switching)**.** *The self-fulfilling equilibria in Khorrami and Mendo (2025) are always recessionary and depend on beliefs about what happens in extreme states when x is very low. Motivated by this, it is natural to consider a fiscal rule that incorporates a state-dependent switch in extreme recessions. Consider, for some threshold $\chi < 0$,*

$$\alpha_t = \begin{cases} \bar{\alpha} > 0, & \text{if } x_t \geq \chi; \\ 0, & \text{if } x_t < \chi. \end{cases} \tag{17}$$

This government does "active fiscal" whenever x_t falls low enough (i.e., in extreme recessions). This somewhat resembles real-world policies: in normal times, governments responsibly pay back debts by raising taxes and/or reducing spending; but in emergencies, governments abandon their fiscal responsibilities in favor of "stimulus" to lift the economy out of crisis.

Can uncertainty traps survive this policy? Consider a sunspot equilibrium with a stationary distribution for $x_t \in (-\infty, x_{max}]$. *Khorrami and Mendo (2025) show how to construct such equilibria. Assume without loss of generality that policy (17) has picked the threshold* χ *such that that fiscal policy is not always active, i.e., they pick* $\chi < x_{max}$. *In that case, we necessarily have* $\mathbb{P}\{x_t \geq \chi\} > 0$. *Using the ergodic theorem, we then have that*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \alpha_t dt \xrightarrow{a.s.} \mathbb{E}[\alpha_t] = \bar{\alpha}\mathbb{P}\{x_t \geq \chi\} > 0.$$

This implies $\int_0^\infty \alpha_t dt = +\infty$, *so government transversality holds by Lemma 1. Consequently, the conjectured sunspot equilibrium survives such a fiscal policy.*

Despite being ultimately a passive policy, the recessionary interventions described by Example 2 are appealing. In the next section, we strengthen Example 2 in two ways: (i) fiscal policy is more "aggressive" in deep recessions; and (ii) the intervention threshold is "adaptive" to ensure regimes switch often enough. This will render the fiscal policy active and thus enable equilibrium selection.

2.2 Fiscal Backstops

Here, we discuss a class of fiscal policies that successfully eliminates uncertainty traps. We refer to these policies as *fiscal backstops* because they are less invasive than an always-active fiscal policy, in the sense that an active fiscal regime only emerges sometimes. The details are as follows.

First, when in the active regime, fiscal policy is sufficiently aggressive. We model α_t as a two-state process, with states $\bar{\alpha} > 0$ and $\underline{\alpha} < 0$. The regime with $\bar{\alpha} > 0$ is the standard passive one. By contrast, $\underline{\alpha} < 0$ captures a fiscal policy regime that is very aggressively active: as debt-to-GDP rises, surpluses decline (e.g., spending rises and/or taxes fall). In some sense, this is beyond irresponsible because it allows debt to spiral out of control. But it reflects some aspects of real-world policies: debt-to-GDP increases are often due to negative GDP shocks to which the government responds by spending even more, as in stimulus packages, which further raises the debt-to-GDP ratio. An important consideration is the size of $\underline{\alpha}$.

Second, we design the intervention threshold in such a way that the aggressively-active regime occurs with positive probability over the long run. To achieve this, we assume that regimes switch at a threshold that is endogenous to the equilibrium being played—we refer to this as an *adaptive backstop* and explain this name shortly. What we specifically assume is that policy can condition its switching point on the stationary distribution in equilibrium. Let $p(x)$ denote the (marginal) stationary density of x_t in equilibrium, and let

$$\chi_q := \inf \left\{ \chi : \int_{-\infty}^{\chi} p(x) dx \geq q \right\} \tag{18}$$

denote the q^{th} percentile of p. For some $q > 0$, an adaptive backstop is characterized by

$$\alpha_t = \begin{cases} \bar{\alpha}, & \text{if } x_t \geq \chi_q; \\ \underline{\alpha}, & \text{if } x_t < \chi_q. \end{cases} \tag{19}$$

The feature of this policy is that χ_q is chosen such that the probability of the active regime is, in every equilibrium, at least q. Notice that α_t is purely a function of x_t at equilibrium. (Technically, the intervention threshold χ_q depends on the endogenous distribution, so it is also a "function of the equilibrium" and solves a fixed point problem. But, as mentioned, *at equilibrium* χ_q is merely a constant, so α_t depends only on x_t.)

Is it realistic for policy to condition on the equilibrium stationary distribution? We think so. Imagine policy promises to intervene in the worst 10% of recessions. To achieve this, policy can start by setting an initial threshold $\tilde{\chi}^{(1)}$. Over time, perhaps policymakers see that sunspot equilibria are persisting and they are intervening less than promised, maybe only 5% of the time. They can promise more intervention in the future by tightening their threshold, $\tilde{\chi}^{(2)} > \tilde{\chi}^{(1)}$. As they observe equilibrium dynamics, policymakers continue to adjust the thresholds $\tilde{\chi}^{(n)}, \tilde{\chi}^{(n+1)}, \dots$ until they converge to something approximating their desired 10th percentile threshold $\chi_{0.10}$. Dynamic adjustment is one way to think about achieving our so-called adaptive backstops.

With the combination of a very aggressive active regime and an adaptive backstop, fiscal policy imposes a non-redundant debt valuation equation on every equilibrium. To see this, recall that an adaptive threshold χ_q means that $\mathbb{P}\{x_t < \chi_q\} \geq q$, regardless of the equilibrium being played. Consequently, we have by the ergodic theorem that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha_t dt \xrightarrow{a.s.} \mathbb{E}[\alpha_t] = \bar{\alpha} \mathbb{P}\{x_t \geq \chi_q\} + \underline{\alpha} \mathbb{P}\{x_t < \chi_q\} \leq (1-q)\bar{\alpha} + q\underline{\alpha}.$$

So long as

$$\underline{\alpha} < -\bar{\alpha} \frac{1-q}{q}, \tag{20}$$

we have $\mathbb{E}[\alpha_t] < 0$ and hence condition (15) of Lemma 2 is satisfied. If policy is either active sufficiently often (i.e., q is high enough) or sufficiently aggressive when active (i.e., $\underline{\alpha}$ is negative enough), we thus obtain a non-redundant debt valuation equation.[3]

In all designs, we require that the active-fiscal regime is paired with a passive-money regime, which is the standard coordination in the literature (Leeper, 1991). Write the interest rate rule as the linear rule with regime-specific coefficients,

$$\Phi(x, \pi; \alpha) = \phi_x(\alpha)x + \phi_\pi(\alpha)\pi, \tag{21}$$

allowing the monetary regime to shift in tandem with the fiscal regime. We assume that

$$\begin{aligned} \text{(Active fiscal / passive money):} \quad & \phi_x(\underline{\alpha}) \leq 0 \quad \text{and} \quad \phi_\pi(\underline{\alpha}) < 1 \\ \text{(Passive fiscal / arbitrary money):} \quad & \phi_x(\bar{\alpha}) \text{ free} \quad \text{and} \quad \phi_\pi(\bar{\alpha}) \text{ free} \end{aligned} \tag{22}$$

Appropriate fiscal-monetary coordination guarantees that our equilibrium selection results do not stem from "inconsistent or overdetermined policies" as critiqued by Cochrane (2011). For the purpose of eliminating sunspot volatility, perhaps surprisingly, we do not need any assumption about monetary policy when fiscal policy is active ($\alpha = \bar{\alpha}$).

We now state a formal result. Under adaptive backstops, there cannot exist any uncertainty trap within the class discovered in Khorrami and Mendo (2025). The result extends to uncertainty traps that can even include a dependence on fiscal shocks Ω.

Theorem 3. *Consider fiscal policy with short-term debt ($\beta = \infty$), with no output-inflation feedbacks ($\gamma(\cdot) = 0$), and with debt feedback α_t satisfying (18)-(19)-(20) and $\bar{\alpha} < \rho$, paired with monetary policy (21)-(22). Within the class of equilibria in which inflation and volatilities take the form $\pi_t = \pi(x_t, \Omega_t)$, $\sigma_{x,t} = \sigma_x(x_t, \Omega_t)$, and $\varsigma_{x,t} = \varsigma_x(x_t, \Omega_t)$, any equilibrium must have zero sunspot volatility, i.e., $\sigma_{x,t} = 0$.*

The equilibrium with fiscal backstops bears many similarities to the conventional "fiscal equilibrium" that emerges under always-active fiscal policy. In particular, the reason $\sigma_{x,t} = 0$ must hold is that demand is anchored by the valuation-like equation, which recall says that real debt-to-GDP is related to the following present-value of surpluses

[3] A natural question is why the backstop needs to be *adaptive*. Imagine the adaptive threshold χ_q were replaced by a fixed threshold χ. The problem: there are stationary sunspot equilibria in which the tail probability $\mathbb{P}\{x_t < \chi\}$ is vanishingly small (this is shown in Khorrami and Mendo, 2025). In these equilibria, fiscal policy would be almost-always passive and provide no discipline to the dynamics. To rule out all sunspot equilibria with a fixed threshold, policy would either need to increase the threshold (i.e., put $\chi = 0$ as in Theorem 2) or explode the level of aggression (i.e., take $\underline{\alpha} \to -\infty$), neither of which is particularly appealing.

(in the case where $\gamma(\cdot) = 0$):

$$b_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} e^{\int_t^u \alpha_z dz} \hat{s}_u du\Big].$$

This not only eliminates sunspot volatility but pins down overall demand volatility as follows. Since $\hat{s}_t$ is an exogenous function of Ω_t, while α_t in our example policies is a function of (Ω_t, x_t), the valuation-like equation implies that b_t is a function of (Ω_t, x_t). From its definition, debt-to-GDP has a sensitivity to fiscal shocks $d\mathcal{Z}_t$ of $-b_t\varsigma_{x,t}$. On the other hand, given we just showed that debt-to-GDP is a function of (Ω_t, x_t), its sensitivity to fiscal shocks must also be $\varsigma_{x,t}\partial_x b_t + \varsigma_{\Omega,t}\partial_\Omega b_t$, by Itô's formula. Equating the two and rearranging for ς_x implies demand inherits fiscal shocks according to

$$\varsigma_x(x,\Omega) = -\frac{\varsigma_\Omega(\Omega)\partial_\Omega b(x,\Omega)}{b(x,\Omega) + \partial_x b(x,\Omega)} \tag{23}$$

Fiscal-based volatility emerges even during regimes in which fiscal policies appears to behave passively. Thus, a general conclusion about fiscal backstops is that they substitute self-fulfilling demand volatility (σ_x) for fiscal volatility (ς_x).

On the other hand, fiscal backstops permit some flexibility which is absent under the always-active fiscal policy. This flexibility is that, as (22) makes clear, monetary policy may take any reaction function during times when a passive fiscal regime emerges. From the perspective of quantitative and empirical researchers, this is an important degree of freedom, since it allows them to pick the monetary rule that best fits the data. From the perspective of theorists, the flexibility allows us to pick an optimal monetary rule unfettered by any equilibrium selection constraints—by contrast, typical linearized analyses often require active money be paired with passive fiscal (e.g., they often require $\phi_x(\underline{\alpha}) > 0$ and $\phi_\pi(\underline{\alpha}) > 1$). If, for example, the best policy involves passive fiscal regimes to be paired with an interest rate peg, that is totally permissible here.

2.3 Long-term debt

One important generalization replaces short-term debt with long-term debt. This is naturally of interest because short-term debt prices can never respond to shocks. This may lead one to think that short-term debt mechanically, in a knife-edge sense, rules out self-fulfilling demand volatility.

Assume the simplest version of active fiscal policy without any debt feedback, $\alpha = 0$, and without fiscal rules, $\gamma(\cdot) = 0$. Then, equation (16) implies that debt-to-GDP is purely a function of exogenous states $b_t = b(\Omega_t)$ and therefore cannot have any sunspot volatility. Equate this expression to its definition $Q_t B_t / P_t Y_t$ and apply Itô's formula to both sides, recalling equation (6) for $\dot{B}_t$ and that $\dot{P}_t / P_t = \pi_t$. By matching the "dZ" terms, we obtain

$$\sigma_{Q,t} = \sigma_{x,t}, \tag{24}$$

where σ_Q denotes the sunspot loading of $\log(Q_t)$ on dZ_t. In other words, the self-fulfilling demand shocks must be absorbed by long-term debt prices. The key question is whether the pricing of long-term debt in (12) is consistent with this absorption.

Now, to price each bond, note that the nominal SDF in this setting is

$$\frac{M_t}{P_t} = \exp\Big[- \int_0^t \iota_u du - \frac{1}{2} \int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u \Big].$$

Using the notation $\tilde{\mathbb{E}}$ for the risk-neutral expectation (which absorbs the martingale $\frac{1}{2}\int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u$), the debt price from (12) is then

$$Q_t = \tilde{\mathbb{E}}_t \Big[\int_t^\infty \beta e^{-\int_t^T (\iota_u + \beta) du} dT \Big].$$

This equation determines the volatility σ_Q, which then determines the sunspot demand volatility σ_x via (24).

To develop an intuition about why demand volatility is eliminated, we first consider the special example where the interest rate is pegged $\iota_t = \bar{\iota}$. If so, then the per-unit bond pricing equation implies $Q_t = \frac{\beta}{\bar{\iota} + \beta}$. Debt prices are constant, so $\sigma_Q = 0$, and therefore equation (24) implies $\sigma_x = 0$. In fact, the risk-neutral bond pricing formula just above reveals that the *only way* self-fulfilling demand can enter Q_t is via the interest rate rule. But this suggests that the result is much more general than the peg example: monetary policy would need to follow a very particular rule in order to create fluctuations in the bond price that are consistent with self-fulfilling demand, which generically would not happen.

With unpegged interest rates, the debt price is no longer constant and can have volatility. However, the volatility implied by the bond pricing equation (12) is inconsistent with the bond price volatility required to support self-fulfilling demand, unless all these volatilities are zero. To summarize the reasoning, the introduction of long-term debt allows for one extra degree of freedom, namely σ_Q, to absorb self-fulfilling demand shocks, but it also introduces an extra constraint, namely the no-arbitrage pricing equation for a single unit of debt. If σ_Q were some arbitrary process absorbing demand shocks, that would violate the pricing equation for debt. For analytical tractability, we prove this result in the special case without fiscal states Ω, although one expects the reasoning to apply much more generally.

Theorem 4. *Consider fiscal policy with long-term debt ($\beta < \infty$), with no output-inflation feedbacks ($\gamma(\cdot) = 0$), and with no debt feedback ($\alpha = 0$). Suppose there are no exogenous fiscal states, so that the surplus-to-GDP ratio is $s_t = \bar{s}$ constant. Within the class of equilibria in which inflation and volatilities take the form $\pi_t = \pi(x_t)$ and $\sigma_{x,t} = \sigma_x(x_t)$, any equilibrium must have zero sunspot volatility, i.e., $\sigma_{x,t} = 0$.*

2.4 Fiscal rules

Our next generalization allows surpluses to respond to endogenous variables, similarly to the interest rate rule. We introduce a fiscal rule $\gamma(x, \pi)$, which is a bounded continuous function that satisfies $\gamma(0, 0) = 0$. For analytical tractability, we assume an active fiscal policy without any debt feedback, $\alpha = 0$, and also assume the absence of fiscal states Ω, so that the exogenous piece of surpluses is given by $\hat{s} = \bar{s} > 0$. We also revert back to the setting with short-term debt.

Recall the debt valuation computation from (11). Recall also that we consider a class of equilibria where π_t and $\sigma_{x,t}$ are purely functions of x_t. In that case, we have the major simplification that $b_t = b(x_t)$ for some function b that only depends on x_t.[4] In that case, even without computing the function b, by applying Itô's formula to (11) and examining the loading on the sunspot shock dZ, we can say that

$$0 = \sigma_{x,t} \left[b(x_t) + b'(x_t) \right] \tag{25}$$

One possibility is $\sigma_x = 0$, which is the natural case we hope to prove. On the other hand, if $\sigma_x \neq 0$, then the present-value of future surpluses needs to inherit any output gap volatility, implying a particular functional form for b, namely $b(x) \propto e^{-x}$. What we show is that this functional form is generically inconsistent with equation (11), which provides a different equation for b, unless inflation $\pi(x)$ and volatility $\sigma_x(x)$ take a particular form. Then, we show that this particular sunspot form, under some conditions on the policy rules, implies unstable dynamics, meaning that $\sigma_x = 0$ must hold.

[4] The basic reasoning is as follows. First, there are no exogenous states, so future surplus-to-GDP s_T only depends on (x_T, π_T) through the rule γ. Second, while (x_T, π_T) is determined by the entire path of $(x_u, \pi_u, \sigma_{x,u}, \sigma_{\pi,u})_{u \in [t,T)}$, an equilibrium which is Markovian in x has the simplifying property that time-t expectations of functions of (x_T, π_T) are purely functions of x_t. This property implies that b_t is solely determined by x_t.

Theorem 5. *Consider fiscal policy with fiscal rules ($\gamma \neq 0$), with short-term debt ($\beta = \infty$), and with no debt feedback ($\alpha = 0$). Suppose there are no exogenous fiscal states, so that the exogenous part of surplus-to-GDP ratio is $\hat{s}_t = \bar{s}$ constant. Suppose monetary policy follows a linear rule $\Phi(x, \pi) = \phi_x x + \phi_\pi \pi$, with passive coefficients $\phi_x \leq 0$ and $\phi_\pi < 1$. Within the class of equilibria in which inflation and volatilities take the form $\pi_t = \pi(x_t)$ and $\sigma_{x,t} = \sigma_x(x_t)$, any equilibrium must have zero sunspot volatility, i.e., $\sigma_{x,t} = 0$.*

3 Conclusion

In this note, we generalize the equilibrium selection results of Khorrami and Mendo (2025) to permit time- and state-dependent fiscal activism, long-term debt, and fiscal rules depending on output and inflation. We continue to find that self-fulfilling uncertainty in NK models is ruled out by active fiscal policy, broadly defined.

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Online Appendix:
Fiscal Policies as Equilibrium Selection in Uncertain Environments

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A General characterization of fiscal policy

Before proving the main results, we need a preliminary characterization of the equilibrium with fiscal policy. We start by deriving the dynamics of output Y_t, the debt price Q_t, the debt-to-GDP ratio $b_t := \frac{Q_t B_t}{P_t Y_t}$, and the scaled present-value of surpluses $\mathbb{E}_t[\int_t^\infty e^{-\rho(u-t)} s_u du]$. These are the right objects to examine because (11) equates b_t to the present-value $\mathbb{E}_t[\int_t^\infty e^{-\rho(u-t)} s_u du]$.

A.1 Forward-looking objects

Output. In this model, output is $Y_t = Y^* e^{x_t}$, where Y^* is the natural level of output. Using (2), and applying Itô's formula, we have that

$$dY_t = Y_t \Big[\iota_t - \pi_t - \rho + \sigma_{x,t}^2 + |\varsigma_{x,t}|^2 \Big] dt + \sigma_{x,t} dZ_t + \varsigma_{x,t} \cdot d\mathcal{Z}_t. \tag{A.1}$$

Debt price. The bond price Q_t has dynamics of the form

$$dQ_t = Q_t \Big[\mu_{Q,t} dt + \sigma_{Q,t} dZ_t + \varsigma_{Q,t} \cdot d\mathcal{Z}_t \Big] \tag{A.2}$$

for some μ_Q, σ_Q, and ς_Q to be determined. Starting from the per-unit bond pricing equation (12), we have that the object

$$e^{-\beta t} \frac{Q_t M_t}{P_t} + \int_0^t \frac{M_u}{P_u} \beta e^{-\beta u} du$$

is a local martingale and has zero drift. Note that, from the consumption FOC $e^{-\rho t} C_t^{-1} = M_t$, the nominal SDF M_t / P_t has dynamics

$$d(M_t/P_t) = -(M_t/P_t) \Big[\iota_t dt + \sigma_{x,t} dZ_t + \varsigma_{x,t} \cdot d\mathcal{Z}_t \Big] \tag{A.3}$$

Then, by applying Itô's formula to the previous expression, and setting the resulting drift to zero, we have

$$\mu_{Q,t} = \beta - \frac{\beta}{Q_t} + \iota_t + \sigma_{x,t}\sigma_{Q,t} + \varsigma_{x,t} \cdot \varsigma_{Q,t} \tag{A.4}$$

This characterizes the drift of Q.

Debt-to-GDP. Next, we derive the dynamics of the real debt-to-GDP ratio $b_t := \frac{Q_t B_t}{P_t Y_t}$. By Itô's formula,

$$db_t = b_t \frac{\dot{B}_t}{B_t} dt - b_t \frac{\dot{P}_t}{P_t} dt - b_t \frac{dY_t}{Y_t} + b_t \frac{d[Y]_t}{Y_t^2} + b_t \frac{dQ_t}{Q_t} - b_t \frac{d[Q, Y]_t}{Q_t Y_t}$$

We then substitute the flow government budget constraint (6) for $\dot{B}_t$, the price level dynamics $\dot{P}_t / P_t = \pi_t$, the dynamics of Y_t from (A.1), and the dynamics of Q_t from (A.2) and (A.4). After doing so, we obtain

$$\begin{aligned} db_t = {} & \frac{1}{P_t Y_t} (\beta B_t - \beta B_t Q_t - P_t S_t) dt - b_t \pi_t dt - b_t \Big(\iota_t - \pi_t - \rho + \sigma_{x,t}^2 + |\varsigma_{x,t}|^2 \Big) dt \\ & + b_t \Big(\sigma_{x,t}^2 + |\varsigma_{x,t}|^2 \Big) dt + b_t \Big(\beta - \frac{\beta}{Q_t} + \iota_t + \sigma_{x,t}\sigma_{Q,t} + \varsigma_{x,t} \cdot \varsigma_{Q,t} \Big) dt \\ & - b_t \Big(\sigma_{x,t} dZ_t + \varsigma_{x,t} \cdot d\mathcal{Z}_t \Big) + b_t \Big(\sigma_{Q,t} dZ_t + \varsigma_{Q,t} \cdot d\mathcal{Z}_t \Big) - b_t \Big(\sigma_{Q,t}\sigma_{x,t} + \varsigma_{Q,t} \cdot \varsigma_{x,t} \Big) dt \end{aligned}$$

Simplifying, this becomes

$$db_t = (\rho b_t - s_t) dt - b_t \Big((\sigma_{x,t} - \sigma_{Q,t}) dZ_t + (\varsigma_{x,t} - \varsigma_{Q,t}) \cdot d\mathcal{Z}_t \Big) \tag{A.5}$$

Scaled PV of surpluses. On the other hand, equation (11) also implies that $b_t = \mathbb{E}_t[\int_t^\infty e^{-\rho(u-t)} s_u du]$. From this definition, we have that

$$e^{-\rho t} b_t + \int_0^t e^{-\rho u} s_u du = \mathbb{E}_t \Big[\int_0^\infty e^{-\rho u} s_u du \Big],$$

which is a local martingale. By the martingale representation theorem, we have that

$$d \Big(e^{-\rho t} b_t + \int_0^t e^{-\rho u} s_u du \Big) = e^{-\rho t} \Big(\sigma_{b,t} dZ_t + \varsigma_{b,t} \cdot d\mathcal{Z}_t \Big)$$

for some $\sigma_{b,t}$ and some $\varsigma_{b,t}$. We also have by applying Itô's formula to the left-hand-side,

$$d\Big(e^{-\rho t}b_t + \int_0^t e^{-\rho u}s_u du\Big) = \Big[-\rho e^{-\rho t}b_t + e^{-\rho t}s_t\Big]dt + e^{-\rho t}db_t$$

Equating these last two results, and rearranging for db_t, we have

$$db_t = (\rho b_t - s_t)dt + \sigma_{b,t}dZ_t + \varsigma_{b,t} \cdot d\mathcal{Z}_t \tag{A.6}$$

Comparing results (A.5) with (A.6), we immediately have the following:

Lemma A.1. *In the setting above,*

$$b_t\sigma_{x,t} = b_t\sigma_{Q,t} - \sigma_{b,t} \tag{A.7}$$

$$b_t\varsigma_{x,t} = b_t\varsigma_{Q,t} - \varsigma_{b,t} \tag{A.8}$$

A.2 Characterization of passive versus active policy (Lemmas 1-2)

Proof of Lemma 1. As a preliminary, we purge surpluses of their debt-feedback component. So let us examine the dynamics of $e^{-\int_0^t(\rho-\alpha_u)du}b_t$. Using Itô's formula and the surplus rule (13), we have

$$d\Big(e^{-\int_0^t(\rho-\alpha_u)du}b_t\Big) = e^{-\int_0^t(\rho-\alpha_u)du}\Big[-(\hat{s}_t+\gamma_t)dt - b_t\Big((\sigma_{x,t}-\sigma_{Q,t})dZ_t + (\varsigma_{x,t}-\varsigma_{Q,t})\cdot d\mathcal{Z}_t\Big)\Big],$$

where $\hat{s}_t = \hat{s}(\Omega_t)$ and $\gamma_t = \gamma(x_t, \pi_t)$. Then,

$$\mathbb{E}_0[e^{-\rho T}b_T] = \mathbb{E}_0\Big[e^{-\int_0^T \alpha_t dt}\Big(b_0 - \int_0^T e^{-\rho t}e^{\int_0^t \alpha_u du}(\hat{s}_t+\gamma_t)dt\Big)\Big] \tag{A.9}$$

From here, we can prove the lemma.

By $\alpha_t \geq 0$, we have $e^{-\int_0^T \alpha_t dt} \leq 1$. Also, recall that $\hat{s}_t$ and γ_t are bounded. These assumptions imply we can use the dominated convergence theorem to take $T \to \infty$ inside the expectation on the right-hand-side of (A.9). Then, using condition (14), we get

$$\lim_{T\to\infty} \mathbb{E}_0\Big[e^{-\int_0^T \alpha_t dt}\Big(b_0 - \int_0^T e^{-\rho t}e^{\int_0^t \alpha_u du}(\hat{s}_t+\gamma_t)dt\Big)\Big]$$
$$= \mathbb{E}_0\Big[e^{-\int_0^\infty \alpha_t dt}\Big(b_0 - \int_0^\infty e^{-\rho t}e^{\int_0^t \alpha_u du}(\hat{s}_t+\gamma_t)dt\Big)\Big] = 0.$$

Therefore, the left-hand-side of (A.9) must also vanish as $T \to \infty$. As shown in equation (10), this is equivalent to the transversality condition (7). □

Proof of Lemma 2. We will prove equation (16) at $t = 0$ since the same argument will hold at any $t > 0$. Similar to the derivation leading to equation (A.9), we can obtain

$$\mathbb{E}_0[e^{-\rho T} b_T A_T] = \mathbb{E}_0[I_T], \tag{A.10}$$
$$\text{where} \quad A_T := e^{\int_0^T \alpha_t dt}$$
$$\text{and} \quad I_T := b_0 - \int_0^T e^{-\rho t} e^{\int_0^t \alpha_u du} (\hat{s}_t + \gamma_t) dt$$

In this compact notation, our aim is to prove that $0 = \mathbb{E}_0[I_\infty]$, where $I_\infty := \lim_{T\to\infty} I_T$ is the pointwise limit of I_T (it will be shown that this limit exists).

To proceed with our proof, we will guess and then verify ex-post that $(e^{-\rho T} b_T A_T)_{T>0}$ is uniformly integrable (UI).

To begin, we prove $\mathbb{E}_0[e^{-\rho T} b_T A_T]$ converges to zero along a subsequence. Given the transversality condition $\lim_{T\to 0} \mathbb{E}_0[e^{-\rho T} b_T] = 0$, there exists a subsequence of times $(T_j)_{j=1}^{\infty}$ with $T_j \to \infty$ such that $e^{-\rho T_j} b_{T_j} \to 0$. Given condition (15), we also have $A_T \to A_\infty < \infty$ and so $A_{T_j} \to A_\infty < \infty$. Combining these conditions, we have that $\lim_{j\to\infty} e^{-\rho T_j} b_{T_j} A_{T_j} = 0$. Given that $(e^{-\rho T} b_T A_T)_{T>0}$ is UI, we can conclude by Vitali's convergence theorem that $\lim_{j\to\infty} \mathbb{E}_0[A_{T_j} e^{-\rho T_j} b_{T_j}] = \mathbb{E}_0[\lim_{j\to\infty} A_{T_j} e^{-\rho T_j} b_{T_j}] = 0$.

Next, we have that $(I_{T_j})_{j>0}$ are UI. Indeed, we have proven $\lim_{j\to\infty} \mathbb{E}_0[A_{T_j} e^{-\rho T_j} b_{T_j}] = 0$, so by (A.10), we have that $\lim_{j\to\infty} \mathbb{E}_0[I_{T_j}] = 0$. This convergence-in-mean implies UI.

Next, we establish that $\lim_{j\to\infty} I_{T_j} = \lim_{T\to\infty} I_T =: I_\infty$ (i.e., convergence of I_T is the same along any subsequence). To do this, start by noting that $\hat{s}_t$ and γ_t are uniformly bounded processes. Hence, $\sup_t \alpha_t < \rho$ implies that $\int_T^\infty e^{-\rho t} e^{\int_0^t \alpha_u du} (\hat{s}_t + \gamma_t) dt$ converges to zero as $T \to \infty$, implying that $I_\infty := b_0 - \int_0^\infty e^{-\rho t} e^{\int_0^t \alpha_u du} (\hat{s}_t + \gamma_t) dt$ is well-defined as the limit.

These facts allow us to conclude. The pointwise convergence $I_T \to I_\infty$ (hence convergence in probability), plus the fact that $(I_{T_j})_{j>0}$ are UI, implies that $\lim_{j\to\infty} \mathbb{E}_0[I_{T_j}] = \mathbb{E}_0[\lim_{j\to\infty} I_{T_j}] = \mathbb{E}_0[I_\infty]$ by Vitali's convergence theorem, as desired. As mentioned at the beginning, this implies that the valuation-like equation (16) holds for all $t > 0$.

Finally, we verify that $(e^{-\rho T} b_T A_T)_{T>0}$ is indeed UI as conjectured. Using equation (16), combined with the assumptions that $\sup_t \alpha_t < \rho$, and that $\hat{s}_t$ and γ_t are uniformly bounded, we immediately have that b_t is also uniformly bounded. Using again that $\sup_t \alpha_t < \rho$, we thus have that $e^{-\rho T} A_T b_T$ is bounded, hence UI. □

A.3 Characterization of active fiscal in a large class of equilibria

We may now prove a general result that applies to all relevant sub-cases. To do so, restrict attention to a class of equilibria in which π_t, $\sigma_{x,t}$, and $\varsigma_{x,t}$ are purely functions of (x, Ω), as defined in Definition 1

Lemma A.2. *Suppose the conditions of Lemma 2 hold. The generalized model above has no (x, Ω)-Markov sunspot equilibria "generically," in the sense that, without imposing Condition 1, the $2 + dim(\mathcal{Z})$ endogenous variables $\pi(x, \Omega)$, $\sigma_x(x, \Omega)$, and $\varsigma_x(x, \Omega)$ have only $dim(\mathcal{Z})$ degrees of freedom whenever $\sigma_x \neq 0$. In particular, if $dim(\mathcal{Z}) = 0$ (no fiscal shocks), then $\pi(x)$ and $\sigma_x(x)$ are pinned down uniquely. Furthermore, regardless of $dim(\mathcal{Z}) = 0$, it must hold that*

$$Q(x, \Omega) = Gb(x, \Omega)e^x, \quad \text{on} \quad \{(x, \Omega) : \sigma_x^2(x, \Omega) \neq 0\}, \tag{A.11}$$

for some constant G.

Lemma A.2 shows that there essentially cannot be sunspot equilibria. The reason is that the objects are severely restricted in their degrees of freedom, which then implies that the resulting dynamics could generically not satisfy the additional non-explosion requirements that are needed. This is seen most transparently in the case without fiscal shocks, because then π and σ_x are pinned down uniquely in a way that, we will show, can definitively not be consistent with non-explosion. As a corollary, the result below displays the uniquely determined functional forms of all the key objects, in the case without fiscal shocks.

Corollary A.1. *Without fiscal state variables (no Ω), the model above requires the following to hold whenever $\sigma_x \neq 0$, where G is some constant:*

$$Q(x) = Gb(x)e^x \tag{A.12}$$

$$b(x) = \frac{G^{-1}\beta e^{-x} - \hat{s} - \gamma(x, \pi(x))}{\beta + \pi(x) + \alpha(x)} \tag{A.13}$$

$$\sigma_x^2(x) = 2\frac{\rho\pi(x) - \kappa f(x) - (\bar{\iota} + \Phi(x, \pi(x)) - \pi(x) - \rho)\pi'(x)}{\pi'(x) + \pi''(x)} \tag{A.14}$$

and

$$(\rho - \alpha(x))b(x) - \hat{s} - \gamma(x, \pi(x)) = \frac{b'(x) + b''(x)}{\pi'(x) + \pi''(x)}(\rho\pi(x) - \kappa f(x)) \tag{A.15}$$

$$= \left[\bar{\iota} + \Phi(x, \pi(x)) - \pi(x) - \rho\right] \frac{b'(x)\pi''(x) - b''(x)\pi'(x)}{\pi'(x) + \pi''(x)}$$

Thus, the objects (Q, b, σ_x^2, π) are all pinned down in an equilibrium with $\sigma_x \neq 0$.

Proof of Lemma A.2 and Corollary A.1. We start by using the (x, Ω)-Markov assumption, which implies all dynamics are fully Markovian in (x_t, Ω_t). Hence, the bond price Q_t and the debt-to-GDP b_t are solely functions of x_t and Ω_t, i.e., $Q_t = Q(x_t, \Omega_t)$ and $b_t = b(x_t, \Omega_t)$ for some functions $Q(\cdot)$ and $b(\cdot)$ to be determined. Indeed, in an (x, Ω)-Markov equilibrium, we have that (x_t, Ω_t) is a bivariate Markov diffusion. Now, recall the bond pricing equation (12), which after plugging in the nominal SDF from (A.3) says

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^u (\iota_\tau + \frac{1}{2}(\sigma_{x,\tau}^2 + |\varsigma_{x,\tau}|^2))d\tau - \int_t^u \sigma_{x,\tau} dZ_\tau - \int_t^u \varsigma_{x,\tau} \cdot d\mathcal{Z}_\tau} \beta e^{-\beta(u-t)} du\Big].$$

Since $\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) = \bar{\iota} + \Phi(x_t, \pi(x_t, \Omega_t))$ is purely a function of (x_t, Ω_t), as are $\sigma_{x,t}$ and $\varsigma_{x,t}$, the bond pricing equation above implies that Q_t is purely a function of (x_t, Ω_t). Similarly, we have that $\hat{s}_t + \gamma_t = \hat{s}(\Omega_t) + \gamma(x_t, \pi_t(x_t, \Omega))$ is solely a function of (x_t, Ω_t). Using Lemma 2, we have that

$$b_t = \mathbb{E}_t\big[\int_t^\infty e^{-\rho(u-t)} e^{\int_t^u \alpha_z dz} (\hat{s}_u + \gamma_u) du\big],$$

and so b_t is a function of (x_t, Ω_t) alone.

Let us define the differential operator $\mathscr{L}$ that acts on C^2 functions g of (x, Ω) by

$$\mathscr{L} g = \Big(\mu_x \partial_x + \mu_\Omega' \partial_\Omega + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} + \frac{1}{2}\mathrm{tr}(\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega'}) + \varsigma_x' \varsigma_\Omega \partial_{\Omega x}\Big) g \tag{A.16}$$

This operator produces drifts of any process which is a function of (x, Ω). Apply Itô's formula to Q and b to obtain (after dropping t subscripts)

$$Q\sigma_Q = \sigma_x \partial_x Q \tag{A.17}$$
$$Q\varsigma_Q = \varsigma_x \partial_x Q + \varsigma_\Omega \partial_\Omega Q \tag{A.18}$$
$$Q\mu_Q = \mathscr{L} Q \tag{A.19}$$
$$\sigma_b = \sigma_x \partial_x b \tag{A.20}$$
$$\varsigma_b = \varsigma_x \partial_x b + \varsigma_\Omega \partial_\Omega b \tag{A.21}$$
$$\mu_b = \mathscr{L} b \tag{A.22}$$

Combining these results with equations (A.4), (A.6), (A.7), and (A.8), we obtain

$$\sigma_x = \sigma_x \partial_x Q/Q - \sigma_x \partial_x b/b \tag{A.23}$$

$$\varsigma_x = \varsigma_x \partial_x Q/Q - \varsigma_x \partial_x b/b + \varsigma_\Omega \partial_\Omega Q/Q - \varsigma_\Omega \partial_\Omega b/b \tag{A.24}$$

and

$$(\beta + \bar{\iota} + \Phi(x,\pi))Q - \beta + \sigma_x^2 \partial_x Q + |\varsigma_x|^2 \partial_x Q + \varsigma_x \cdot \varsigma_\Omega \partial_\Omega Q = \mathscr{L}Q \tag{A.25}$$

$$\rho b - \hat{s}(\Omega) - \gamma(x,\pi) - \alpha(x,\Omega) b = \mathscr{L}b \tag{A.26}$$

(Note that in the short-term debt case, which can be derived by taking $\beta \to \infty$, equation (A.25) implies $Q \to 1$ uniformly. In addition, after taking this limit we have $\lim_{\beta\to\infty} \partial_x Q = 0$ and $\lim_{\beta\to\infty} \partial_\Omega Q = 0$, and so $\lim_{\beta\to\infty}(\frac{\beta}{Q} - \beta) = \bar{\iota} + \Phi(x,\pi)$. This limiting result is also consistent with taking the $\beta \to \infty$ in the flow budget constraint (6) in order to recover the conventional budget constraint with short-term debt.)

Now, suppose $\sigma_x \neq 0$. In that case, equation (A.23) says that $1 = \partial_x Q/Q - \partial_x b/b$, and equation (A.24) says that $\partial_\Omega Q/Q = \partial_\Omega b/b$. The first equation implies that $Q(x,\Omega) = b(x,\Omega)G(\Omega)e^x$ for some function $G(\cdot)$. The second equation implies that $G(\Omega) = G$ constant. Thus, equation (A.11) holds. Note that then G is pinned down by equation (11) at time $t = 0$, since combining that equation with (A.11) says $\frac{B_0}{P_0}G = Y^*$. Thus, (A.11) pins down Q given b. Substitute (A.11) into equation (A.25) and then subtract equation (A.26) to get

$$\frac{\hat{s}(\Omega) + \gamma(x,\pi) + \alpha(x,\Omega)b}{b} - \rho + \beta + \bar{\iota} + \Phi(x,\pi) - \frac{\beta}{Gb}e^{-x} = \mu_x - \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)$$

Now, plug in μ_x from the IS curve (2) to get

$$0 = \pi + \alpha(x,\Omega) + \frac{\hat{s}(\Omega) + \gamma(x,\pi)}{b} + \beta - \frac{\beta}{Gb}e^{-x} \tag{A.27}$$

Equation (A.27) thus pins down b given π. Since we only used so far the difference between equations (A.25) and (A.26), we still need to ensure that each one of them holds in isolation. Thus, equation (A.26) also must hold after plugging in μ_x from (2):

$$\begin{aligned}
&\rho b - \hat{s}(\Omega) - \gamma(x,\pi) - \alpha(x,\Omega) b \qquad \text{(A.28)}\\
&\quad = \Big[\bar{\iota} + \Phi(x,\pi) - \pi - \rho + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\Big]\partial_x b + \mu_\Omega' \partial_\Omega b + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} b \\
&\quad + \frac{1}{2}\mathrm{tr}(\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega'} b) + \varsigma_x' \varsigma_\Omega \partial_{\Omega x} b
\end{aligned}$$

Given $(\pi, \sigma_x, \varsigma_x)$, equation (A.28) is a PDE for b. Finally, recall the Phillips curve (3), apply Itô's formula to a generic inflation function $\pi(x,\Omega)$ to replace μ_π, and then plug in μ_x from (2):

$$\begin{aligned}
&\rho\pi - \kappa f(x) \qquad \text{(A.29)}\\
&\quad = \Big[\bar{\iota} + \Phi(x,\pi) - \pi - \rho + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\Big]\partial_x \pi + \mu_\Omega' \partial_\Omega \pi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} \pi \\
&\quad + \frac{1}{2}\mathrm{tr}(\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega'} \pi) + \varsigma_x' \varsigma_\Omega \partial_{\Omega x} \pi
\end{aligned}$$

where $f(x) := \frac{e^{(1+\varphi)x} - 1}{1+\varphi}$ if we are using a nonlinear Phillips curve and $f(x) = x$ if we are using a linearized Phillips curve. Given (σ_x, ς_x), equation (A.29) is a PDE for π. At this point, consider the following experiment. Suppose $\pi(x,\Omega)$ is any function. Then, equation (A.27) pins down $b(x,\Omega)$ uniquely, and equation (A.11) pins down $Q(x,\Omega)$ uniquely. Given π and b, we can compute all their derivatives, and so equations (A.28) and (A.29) pin down 2 dimensions of the $1 + \dim(\mathcal{Z})$ dimensional vector (σ_x, ς_x). In other words, we must pick σ_x and/or ς_x in order to ensure equations (A.28) and (A.29) hold.

Thus, if $\dim(\mathcal{Z}) = 0$, then either $\sigma_x = 0$, or $\pi(x)$ and $\sigma_x^2(x)$ must take a particular form. Note also that these functions are independent of Ω since there are no surplus shocks (hence Ω is not a state variable for any object in the case $\dim(\mathcal{Z}) = 0$). The equations for Corollary A.1 are obtained by removing Ω from all the equations above and rearranging. □

B Proofs of Theorems in Text

Proof of Theorem 3. We proceed in steps. First, we verify the sufficient conditions for "active fiscal" to obtain a valuation equation for b_t. Second, assuming a non-explosive sunspot equilibrium, we obtain the unique solution for b_t in the Markovian environment assumed. Third, we use this solution to pin down inflation and volatility uniquely.

Fourth, we demonstrate that this unique solution cannot be consistent with "stability," namely the non-explosive Condition 1 is violated, a contradiction to the assumption that sunspot volatility is present.

Step 1: valuation equation. First, by the definition of χ_q in (18), every equilibrium satisfies $\mathbb{P}\{x_t < \chi_q\} \geq q$. Given that condition (20) holds and that $\bar{\alpha} < \rho$, we have that the assumptions of Lemma 2 hold, so we obtain valuation equation (16) for b_t.

Step 2: solution for b. We assume a non-explosive sunspot equilibrium. Let the ergodic set of the sunspot equilibrium be $\mathcal{X} := \{(x, \Omega) : x \in (x_{min}(\Omega), x_{max}(\Omega))\}$, where

$$x_{min}(\Omega) := \inf\{x : \sigma_x^2(x, \Omega) > 0\}$$
$$x_{max}(\Omega) := \sup\{x : \sigma_x^2(x, \Omega) > 0\}.$$

Define $\mathcal{X}^\circ := \{(x, \Omega) : \sigma_x^2(x, \Omega) > 0\}$ to be the sub-domain with sunspot volatility. Notice that all (x, Ω) close enough to the boundary of $\mathcal{X}$ lie inside $\mathcal{X}^\circ$.

Now, since π_t, $\sigma_{x,t}$, and $\varsigma_{x,t}$ are functions of (x_t, Ω_t), we have that (x_t, Ω_t) is a Markov process, and so the equilibrium satisfies Definition 1. Since α_t is also a function of (x_t, Ω_t), equation (16) implies that $b_t = b(x_t, \Omega_t)$ for some function b. Now, we apply equation (A.11), noting that with short-term debt we have $Q = 1$. In that case, we then know that for some constant $\bar{b}$,

$$b(x, \Omega) = \bar{b}e^{-x}, \quad \text{on } \mathcal{X}^\circ. \tag{B.1}$$

Step 3a: pinning down π, σ_x^2. Substituting (B.1) into the differential equation (A.28) and rearranging, we have that

$$\phi_x(\alpha)x + (\phi_\pi(\alpha) - 1)\pi = \frac{\hat{s}(\Omega)}{\bar{b}}e^x + \alpha - \rho, \quad \text{on } \mathcal{X}^\circ. \tag{B.2}$$

This equation cannot hold for all $x \in \mathcal{X}^\circ$, except if inflation takes a particular knife-edge functional form—let this solution for inflation be denoted by

$$\pi_0(x, \Omega) := \frac{\hat{s}(\Omega)e^x/\bar{b} + \alpha - \rho - \phi_x(\alpha)x}{\phi_\pi(\alpha) - 1}. \tag{B.3}$$

On the other hand, the function π_0 must also be consistent with the Phillips curve

(3). Applying Itô's formula to π_0, the result is

$$\rho\pi_0 - \kappa x = \Big[\frac{\hat{s}}{\bar{b}}e^x + \alpha - \rho + \frac{1}{2}\sigma_x^2 + \frac{1}{2}|\varsigma_x|^2\Big]\partial_x\pi_0 + \frac{1}{2}\big(\sigma_x^2 + |\varsigma_x|^2\big)\partial_{xx}\pi_0 + \mu_\Omega'\partial_\Omega\pi_0 + \frac{1}{2}\text{trace}[\varsigma_\Omega\varsigma_\Omega'(\partial_{\Omega\Omega'}\pi_0)] + \varsigma_x\varsigma_\Omega'\partial_{x\Omega}\pi_0, \quad \text{on } \mathcal{X}^\circ. \tag{B.4}$$

Given that π_0 is determined, this equation pins down $\frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2) = \Sigma_0$ on $\mathcal{X}^\circ$, where

$$\Sigma_0 := \frac{\rho\pi_0 - \kappa x - \big(\frac{\hat{s}}{\bar{b}}e^x + \alpha - \rho\big)\partial_x\pi_0 - \big(\mu_\Omega'\partial_\Omega\pi_0 + \frac{1}{2}\text{trace}[\varsigma_\Omega\varsigma_\Omega'(\partial_{\Omega\Omega'}\pi_0)] + \varsigma_x\varsigma_\Omega'\partial_{x\Omega}\pi_0\big)}{\partial_x\pi_0 + \partial_{xx}\pi_0} \tag{B.5}$$

Step 3b: useful properties of π_0, Σ_0. We note, for later, some key properties. First, π_0 and its derivatives are finite for all x finite. Second, its limiting values are given by

$$\lim_{x\to-\infty}\pi_0 = \begin{cases} +\infty, & \text{if } \phi_x(\underline{\alpha}) < 0; \\ \frac{\underline{\alpha}-\rho}{\phi_\pi(\underline{\alpha})-1}, & \text{if } \phi_x(\underline{\alpha}) = 0. \end{cases}$$

$$\lim_{x\to-\infty}\partial_x\pi_0 = -\frac{\phi_x(\underline{\alpha})}{\phi_\pi(\underline{\alpha}) - 1} \le 0$$

$$\lim_{x\to-\infty}\partial_{xx}\pi_0 = 0$$

Here, we have used the key fact that $\alpha(x) \to \underline{\alpha}$ as $x \to -\infty$, by definition of the intervention threshold χ_q. Using these properties for π_0, the total diffusion Σ_0 inherits the following properties:

(P1) $\lim_{x\to-\infty}\Sigma_0 \neq 0$.

[Proof: Because of the fact that π_0 and its derivatives are bounded for all finite x, we have that $\Sigma_0 = 0$ if and only if the numerator in (B.5) vanishes. Plugging in the expression for π_0 from (B.3), and then taking the limit $x \to -\infty$ in the numerator of (B.5), we find that $\Sigma_0 \not\to 0$ unless $\phi_x(\underline{\alpha}) = -\rho$ and $\rho\phi_x(\underline{\alpha}) + \kappa(\phi_\pi(\underline{\alpha}) - 1) = 0$. But these conditions cannot hold under monetary policy (22), since $\phi_\pi(\underline{\alpha}) < 1$.]

(P2) $\lim_{x\to-\infty}\Sigma_0 < +\infty$ for all Ω if $\phi_x(\underline{\alpha}) \neq 0$ and on $\{\Omega : \hat{s}(\Omega) > 0\}$ if $\phi_x(\underline{\alpha}) = 0$.

[Proof: Indeed, by the fact that π_0 and its derivatives are finite for all finite x, Σ_0 can only explode if $x \to -\infty$ and if the following condition holds:

$$\lim_{x\to-\infty}\Sigma_0 = +\infty \iff \lim_{x\to-\infty}\frac{\rho\pi_0 - \kappa x}{\partial_x\pi_0 + \partial_{xx}\pi_0} = +\infty \tag{B.6}$$

If $\phi_x(\underline{\alpha}) \neq 0$, (B.6) is equivalent to $\lim_{x\to-\infty}[\rho + \frac{\kappa(\phi_\pi(\underline{\alpha})-1)}{\phi_x(\underline{\alpha})}]x = +\infty$, which cannot hold by the fact that monetary policy (22) uses $\phi_\pi(\underline{\alpha}) < 1$. If $\phi_x(\underline{\alpha}) = 0$, condition (B.6) is equivalent to $\lim_{x\to-\infty} \frac{\bar{b}\kappa(\phi_\pi(\underline{\alpha})-1)x}{\hat{s}(\Omega)e^x} = -\infty$, which cannot hold for any Ω such that $\hat{s}(\Omega) > 0$, again by the fact that monetary policy (22) uses $\phi_\pi(\underline{\alpha}) < 1$.]

Step 4: explosiveness of dynamics. Under the restriction (B.2), the drift of x is given by

$$\begin{aligned} \mu_x &:= \Phi(x, \pi; \alpha) - \pi + \frac{1}{2}\sigma_x^2 + \frac{1}{2}|\varsigma_x|^2 \\ &= \hat{s}e^x/\bar{b} + \alpha - \rho + \Sigma_0, \quad \text{on } \mathcal{X}^\circ. \end{aligned} \tag{B.7}$$

For the domain to be valid as part of an equilibrium, we must have that the dynamics are such that $(x_t, \Omega) \in \mathcal{X}$ forever—we refer to this as "stability." Recall that all points near the boundary of $\mathcal{X}$ are inside $\mathcal{X}^\circ$, and so the drift above is the relevant expression near the boundaries. There are two possibilities which require different analyses: $x_{min}(\Omega) > -\infty$ and $x_{min}(\Omega) = -\infty$.

If $x_{min}(\Omega) > -\infty$, stability requires that the volatilities vanish there, because the drift μ_x in (B.7) is also finite there. So we require $\Sigma_0(x_{min}(\Omega), \Omega) = 0$. (This restricts the boundary $x_{min}(\Omega)$, meaning it cannot be arbitrary; however, this is not critical for the argument below.) Furthermore, stability requires the drift to point "inwards" into the domain $\mathcal{X}$, i.e., $\mu_x(x_{min}(\Omega), \Omega) \geq 0 \geq \mu_x(x_{max}(\Omega), \Omega)$ for all Ω. However, given $\alpha(x)$ defined in (19) is weakly increasing in x, we have the following for all Ω such that $\hat{s}(\Omega) > 0$:

$$\begin{aligned} \mu_x(x_{min}(\Omega), \Omega) &= e^{x_{min}(\Omega)}\hat{s}(\Omega)/\bar{b} + \alpha(x_{min}(\Omega)) - \rho \\ &< e^{x_{max}(\Omega)}\hat{s}(\Omega)/\bar{b} + \alpha(x_{max}(\Omega)) - \rho \\ &\leq e^{x_{max}(\Omega)}\hat{s}(\Omega)/\bar{b} + \alpha(x_{max}(\Omega)) - \rho + \Sigma_0(x_{max}(\Omega), \Omega) = \mu_x(x_{max}(\Omega), \Omega), \end{aligned}$$

which contradicts the stability requirement on the drifts.

On the other hand, if $x_{min}(\Omega) = -\infty$, then stability requires that volatilities explode asymptotically. Indeed, property (P1) above shows that $\lim_{x\to-\infty} \Sigma_0 \neq 0$. If $\lim_{x\to-\infty} \Sigma_0$ were a finite value, then the drift μ_x in (B.7) would also be finite, and so the process x_t would hit $-\infty$ in finite time with positive probability. Thus, the only way to obtain a non-explosive solution is $\lim_{x\to-\infty} \Sigma_0 = +\infty$. Property (P2) above rules this out.

Thus, we have shown that, regardless of how x_{min} is constructed, the dynamics cannot be non-explosive. This contradiction completes the proof. □

Proof of Theorem 4. The assumptions listed imply a constant surplus-to-output ratio $s_t = \bar{s}$. Using equation (11), this implies that $b_t = \bar{s}/\rho$ is constant for any $\pi(x)$ and any $\sigma_x(x)$ functions. We then specialize the results of Corollary A.1 as follows. Using the result for $b(x) = \bar{s}/\rho$, equation (A.13) then pins down inflation as

$$\pi(x) = \frac{\rho\beta}{G\bar{s}}e^{-x} - \beta - \rho, \quad \text{when} \quad \sigma_x \neq 0. \tag{B.8}$$

Note that $\pi'(x) + \pi''(x) = 0$. Then, equation (A.14) implies that, after plugging in the derivatives of π from (B.8),

$$e^{-x}\frac{\rho\beta}{G\bar{s}}\left(\bar{\imath} + \Phi(x,\pi) - \pi\right) - \kappa f(x) = \rho(\rho+\beta), \quad \text{when} \quad \sigma_x \neq 0. \tag{B.9}$$

But everything is pinned down in equation (B.9). The result cannot be consistent with the solution for π in (B.8) unless the monetary policy rule Φ takes a knife-edge form, and so generically we reach a contradiction. Thus, $\sigma_x = 0$ must hold. □

Proof of Theorem 5. We specialize the results of Corollary A.1 as follows. Using short-term debt ($\beta \to \infty$) in equation (A.13) implies that

$$b(x) = \bar{b}e^{-x}, \quad \text{when} \quad \sigma_x \neq 0,$$

for $\bar{b} = 1/G$. Notice that $b'(x) + b''(x) = 0$ in this solution. Thus, equation (A.15), after plugging in the solution for b and its derivatives, says that

$$\bar{s} + \gamma(x,\pi) = (\bar{\imath} + \Phi(x,\pi) - \pi)\bar{b}e^{-x}, \quad \text{when} \quad \sigma_x \neq 0. \tag{B.10}$$

Equation (B.10) pins down π uniquely when $\sigma_x \neq 0$, unless the rules $\gamma(\cdot), \Phi(\cdot)$ take a knife-edge form. Finally, equation (A.14) specializes to

$$\sigma_x^2 = \tilde{\sigma}_x^2 := 2\frac{\rho\pi - \kappa f(x) - [\bar{\imath} + \Phi(x,\pi) - \pi - \rho]\pi'}{\pi' + \pi''}, \quad \text{when} \quad \sigma_x \neq 0. \tag{B.11}$$

Given the solution for π, this pins down σ_x^2 uniquely when it is non-zero.

Given the functions $b(x)$, $\pi(x)$, and $\sigma_x^2(x)$ are all pinned down assuming $\sigma_x \neq 0$, it remains to verify that the candidate sunspot equilibrium explodes, which then implies $\sigma_x = 0$. This part of the proof follows a very similar argument to Theorem 3 and therefore we omit it. □